FURTHER PHYSICS

VOLUME 2

•SECOND EDITION•

FUNG, SUN & YOUNG

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•SECOND EDITION•

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Preface

Further Physics is written for students taking the Advanced Level Physics course in Hong Kong. With careful selection of the text material, it can also be used for the Advanced Supplementary Level. It is published in two volumes:

Volume 1 Introduction
Mechanics
Wave motion

Volume 2 Fields, electricity and electromagnetism
Matter

Although self-contained, *Further Physics* is best used together with the following titles:

Further Physics Experimental Workbook, by P. Sun

Practical Physics for the HKALE, by A. Li and P. Sun

The three titles constitute a full set of physics books published by Longman for the Advanced Level. The progress in *Further Physics* and that in the *Experimental Workbook* are closely parallel with frequent cross-references. In general, experimental details will be found in the workbook, whereas the integration of experimental findings into a conceptual framework will be given emphasis in the textbook.

The content in the textbook is closely adhered to the examination syllabus in the sense that all the prescribed topics are covered, and to the necessary depth. However, in the firm belief that nothing is more important than a sure and thorough grasp of fundamental concepts, we have treated a few topics to somewhat greater depth than is likely to be required for examination purposes. Additional topics of interest are included. A few topics provide background for practical work. In such cases, reference is made to *Practical Physics for the HKALE* for the detailed experimental set-up and procedure.

It is not intended that teachers should lecture on every topic covered. Students at this level should develop the ability and habit of learning on their own. For example, the two introductory chapters, sections reviewing lower form work, material outside the examination syllabus as well as material of a largely factual nature can all be assigned as home reading. For other topics, teachers may like to follow the text more closely. It must be realised that students should be selective in what they read, and we hope that different students can derive different benefits and insights from this textbook.

One outstanding feature of the Advanced Level Physics Examination is the *Written Practical* of Paper III. This is of great value from an educational point of view precisely because there is no set syllabus, thus focusing upon understanding rather than factual knowledge. Not surprisingly, many students do not feel confident about this part of the examination and would welcome opportunities for practice. Accordingly, a number of such exercises are included. For the sake of greater interest and sense of realism, several of these exercises are based on simplified versions of research work carried out here in Hong Kong, and we thank our friends and colleagues who made available the details of their work, in particular Drs. C.L. Choy, A.F. Leung and W.P. Leung. These exercises provide occasions to learn about experimental design and data analysis, and to broaden horizons in experimental physics. However, it should be emphasised that actual experimental skills can only be acquired through practice in the laboratory.

As mentioned before, this text is also written for the Advanced Supplementary Level. The required material is in fact a subset of that for the Advanced Level, and students can find their way easily enough.

We welcome any suggestions for improvement. Last but not least, our thanks are due to Mr. S.K. To for drawing all the diagrams in this book.

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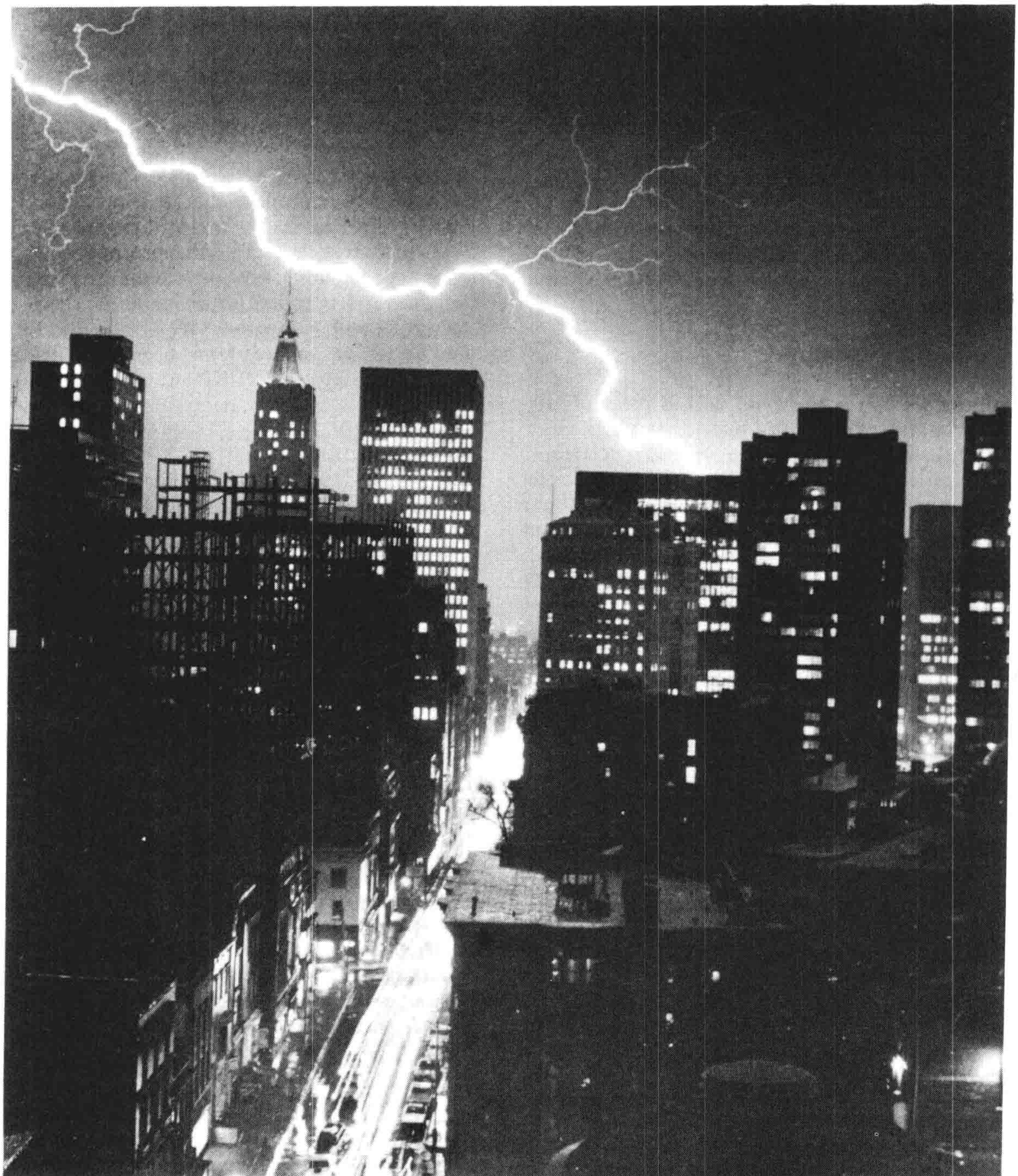
Contents

Fields, electricity and electromagnetism



GRAVITATION

Motion is determined by forces, many of which act between objects in contact with each other — a push or a pull, friction, tension in a string, etc. However, fundamentally, all forces belong to one of the following four types:

(a) Gravitational force,
(b) Electric and magnetic forces,
(c) Weak force responsible for beta decay,
(d) Strong force which holds the nucleus together.

All these forces act between objects which are not necessarily in contact, i.e. they may act over a distance. Gravity will be discussed in this chapter and electricity and magnetism will be discussed in later chapters. The weak and strong forces will only be touched upon very briefly later in the course.

17.1 Concept of field

A mass m released near the earth's surface falls with an acceleration $g = 9.8\,\mathrm{m\,s^{-2}}$. According to Newton's second law, the mass must experience a downward gravitational force mg. Similarly, a charge q placed near another charge experiences an electric force; a current-carrying wire placed near a magnet will experience a magnetic force. We may say that the mass m is inside a gravitational field, the charge q is inside an electric field and the current-carrying wire is inside a magnetic field. A **field** (場) is simply a region where a force is experienced by a suitable test object.

In particular, a **gravitational field** (引力場) is a region where a gravitational force is experienced by a test mass. The force must be proportional to the mass m because all objects fall at the same rate, provided air resistance is negligible. Thus the **intensity of the gravitational field**, also called the **gravitational field strength** (引力場强度), is defined as the force per unit mass.

$$\text{Gravitational field strength} = \frac{\text{gravitational force}}{\text{mass}}$$

In other words, the gravitational field strength is $\vec{F}/m = \vec{g}$, which has a magnitude of $9.8\,\mathrm{N\,kg^{-1}}$ near the earth's surface. Note that $\mathrm{N\,kg^{-1}}$ is the same as $\mathrm{m\,s^{-2}}$:

$$1\ \mathrm{N\,kg^{-1}} = 1\,(\mathrm{kg\,m\,s^{-2}})\,\mathrm{kg^{-1}} = 1\ \mathrm{m\,s^{-2}}$$

Also, $\vec{g}$ should be regarded as a vector in the same direction as the force, i.e. downward. Note the following features of the field strength.

(a) Although the field strength is exactly the same as the 'acceleration due to gravity', the new terminology emphasises that g is present even when the object is *not* accelerating (because there are other forces).
(b) If a mass m is placed near the earth's surface, it is subjected to a force mg; if the mass is removed, there is no force. But we regard g as being always present; it is simply a property of that point in space.
(c) The value of g may vary from point to point, e.g. it is smaller at points far from the earth. It may even vary from time to time. For example, for a fixed point in outer space, g may be very large when a planet passes by, but becomes very small when the planet has moved away.

Although the most common example of a gravitational field is that due to the earth, there will of course be such fields due to other objects as well.

We now mention a few points of notation.

(a) We shall often use g ($= 9.8\,\mathrm{N\,kg^{-1}}$) to denote the gravitational field strength near the earth's surface. The value elsewhere will often be denoted as g' to avoid confusion.
(b) If we choose the upward direction as positive, then g should be $-9.8\,\mathrm{N\,kg^{-1}}$. However, the direction of the gravitational force is always obvious, so we shall usually omit the sign and only write the magnitude of g or the gravitational force F.

(c) Physicists often use the term 'field' to mean 'intensity of the field', i.e. the 'field strength'. This is the modern usage.

17.2 Inverse square law and circular orbits

It is obvious that the gravitational force F between two objects A and B is attractive. The force on A is directed towards B and vice versa. What about the magnitude of the force?

It is also obvious that F becomes weak if the two objects are far apart. How does F depend on the separation r? This section discusses this force and also circular motion under the influence of this force. More complicated motion (e.g. elliptical orbits) will be discussed in Section 17.6.

Motion of the moon

Recall the following experiment. A mass m attached to the end of a string is swung in a circle of radius r (Fig. 17.1). It is observed that the circular motion has a period T. Clearly there must be a centripetal tension F to keep the mass in circular motion:

$$F = ma = m\frac{v^2}{r} = m\frac{(2\pi r/T)^2}{r}$$

$$\frac{F}{m} = 4\pi^2\frac{r}{T^2} \qquad (1)$$

Thus the tension force per unit mass can be found from the radius and period of motion.

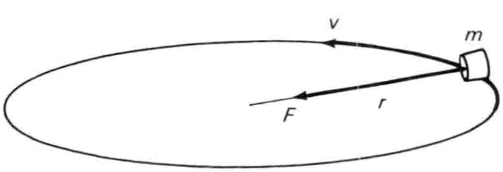


Fig. 17.1

In exactly the same way, the moon is moving in a circle of radius $r = 3.84 \times 10^8$ m around the earth, with a period $T = 2.36 \times 10^6$ s (27.3 d). The centripetal pulling force is provided by the gravitational attraction of the earth (Fig. 17.2). The force per unit mass due to the earth is found using Eq.(1); we denote the result as g':

$$g' = 4\pi^2 \times \frac{3.84 \times 10^8 \text{ m}}{(2.36 \times 10^6 \text{ s})^2}$$

$$= 2.72 \times 10^{-3} \text{ N kg}^{-1}$$

Compared to the value $g = 9.8$ N kg^{-1} on the surface of the earth,

$$\frac{g'}{g} = 2.78 \times 10^{-4} \approx \frac{1}{3600}$$

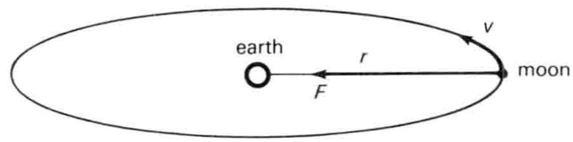


Fig. 17.2

The usual value of g refers to a point whose distance from the centre of the earth is just the radius R of the earth, $R = 6.38 \times 10^6$ m. Let us assume that g' varies as an inverse power of the distance r from the centre of the earth:

$$g \propto r^{-n}$$

The minus sign means that g decreases when r increases. Then

$$\frac{g'}{g} = \left(\frac{r}{R}\right)^{-n} = \left(\frac{3.84 \times 10^8 \text{ m}}{6.38 \times 10^6 \text{ m}}\right)^{-n} \approx 60^{-n}$$

It is easily seen that $n = 2$, i.e. the gravitational force varies as $1/r^2$.

Planets

Even more convincing evidence can be obtained by considering the motion of planets around the sun. All the planets move in nearly circular orbits. (See Section 17.6 for a more accurate description.) Let the mass of a planet be m, its distance from the sun be r and the period of motion be T. Assume that the force per unit mass due to the gravitational attraction of the sun varies as an inverse power of r:

$$\frac{F}{m} = \frac{k}{r^n} \qquad (2)$$

where k is a proportionality constant which is the same for all planets in the *same* solar system. From Eqs. (1) and (2)

$$4\pi^2 \frac{r}{T^2} = \frac{k}{r^n}$$

$$\boxed{\frac{r^{n+1}}{T^2} = \frac{k}{4\pi^2}} \quad (3)$$

Thus the ratio r^{n+1}/T^2 should be the same for all planets in the same solar system. From the known orbit radii and periods of motion of the planets (Table 17.1), it is found that the values r^3/T^2 are the same. Thus $n = 2$, i.e. $F \propto 1/r^2$.

Planet	Orbit radius r/AU	Period T/y	$\frac{r^3}{T^2}$/AU^3 y^{-2}
Mercury 水星	0.387	0.241	0.998
Venus 金星	0.723	0.615	0.999
Earth 地球	1.000	1.000	1.000
Mars 火星	1.524	1.881	1.000
Jupiter 木星	5.203	11.862	1.001
Saturn 土星	9.539	29.458	1.000
Uranus 天王星	19.182	84.013	1.000
Neptune 海王星	30.058	164.793	1.000
Pluto 冥王星	39.439	243.686	1.000

Table 17.1 Orbit radii and periods of motion of planets

In astronomy, it is convenient to express distances in astronomical units (AU), defined as the mean distance of the earth from the sun:

$$1 \text{ AU} = 1.495 \times 10^{11} \text{ m}$$

and the time in years. Naturally, for the earth, $r^3/T^2 = 1$ AU^3 y^{-2} *exactly*. Thus the constant k for our solar system is

$$\begin{aligned} k &= 4\pi^2 \text{ AU}^3 \text{ y}^{-2} \\ &= 4\pi^2 (1.495 \times 10^{11} \text{ m})^3 (365 \times 24 \times 3600 \text{ s})^{-2} \\ &= 1.326 \times 10^{20} \text{ m}^3 \text{s}^{-2} \end{aligned}$$

The fact that r^3/T^2 is the same for all planets in our solar system was first observed by Kepler, and is known as Kepler's third law of planetary motion. (See Section 17.6.)

Law of gravitation

Consider the gravitational force F between an object of mass M (say the sun) and another object of mass m (say the earth) at a distance r apart. This force acts on and is proportional to m. According to Newton's third law, an equal force acts on M, so the force should also be proportional to M. Moreover, we have just shown that $F \propto 1/r^2$. Putting all these together:

$$\boxed{F = G\frac{Mm}{r^2}} \quad (4)$$

where the proportionality constant G is called the universal gravitational constant. Eq. (4) — Newton's inverse square law of universal gravitation — only gives the *magnitude* of F. The direction is of course attractive. Thus if the radially outward direction is regarded as positive, then F should have a *minus* sign.

Let M be the sun and m be any planet. Then by comparing Eq. (2) (with $n = 2$) with Eq. (4), we see

$$GM = k = 1.326 \times 10^{20} \text{ m}^3 \text{s}^{-2} \quad \text{(Sun)}$$

By considering planetary motion alone, we cannot separate G and M. Cavendish first determined the value of G in 1798 by measuring the tiny gravitational force between two metal balls. Such experiments give

$$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$$

The mass of the sun can then be deduced:

$$\begin{aligned} M = \frac{k}{G} &= \frac{1.326 \times 10^{20} \text{ m}^3 \text{ s}^{-2}}{6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}} \\ &= 1.99 \times 10^{30} \text{ kg} \quad \text{(Sun)} \end{aligned}$$

Note that the value of G is universal. However the value of k (and M) applies only to the sun.

Mass and density of the earth

Knowing G, we can also find the mass of the earth M. A mass m on the surface of the earth is subjected to a force

$$F = G\frac{Mm}{R^2}$$

where R is the radius of the earth. But this must be equal to mg, so

$$\boxed{g = \frac{GM}{R^2}} \quad (5)$$

Hence we can calculate the mass of the earth to be

$$M = \frac{gR^2}{G} = \frac{(9.8\ \mathrm{m\,s^{-2}})(6.38 \times 10^6\ \mathrm{m})^2}{6.67 \times 10^{-11}\ \mathrm{kg}}$$

$$= 5.98 \times 10^{24}\ \mathrm{kg} \qquad \text{(Earth)}$$

We can also determine the value of M by considering the lunar orbit. (See Problem 4.)

The average density of the earth is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3g}{4\pi RG}$$

$$= \frac{3 \times 9.8\ \mathrm{m\,s^{-2}}}{4\pi \times 6.38 \times 10^6\mathrm{m} \times 6.67 \times 10^{-11}\mathrm{N\,m^2\,kg^{-2}}}$$

$$= 5.5 \times 10^3\ \mathrm{kg\,m^{-3}}$$

i.e. 5.5 times the density of water. The masses and average densities of the other planets are given in Table 17.2.

Planet	Mass/kg	Average density/10^3 kg m^{-3}
Mercury	3.33×10^{23}	5.46
Venus	4.87×10^{24}	5.23
Earth	5.98×10^{24}	5.52
Mars	6.42×10^{23}	3.92
Jupiter	1.90×10^{27}	1.31
Saturn	5.68×10^{26}	0.70
Uranus	8.72×10^{25}	1.3
Neptune	1.02×10^{26}	1.66
Pluto	6.6×10^{23}	4.9

Table 17.2 Masses and densities of planets

Circular orbits

We now regard the inverse square law as known and summarise the physics of circular motion of a small mass m ('planet') around a large mass M ('sun'). From the force law and Newton's second law

$$\frac{GMm}{r^2} = ma = m\frac{v^2}{r} = m\frac{(2\pi r/T)^2}{r}$$

$$\boxed{\frac{r^3}{T^2} = \frac{GM}{4\pi^2}} \qquad (6)$$

Now consider satellites moving around the earth. Suppose a satellite is in an *equatorial* orbit and has a period of exactly one day (Fig. 17.3). The earth itself also rotates once a day. So the satellite always stays above the same spot on earth if the two are rotating in the *same* direction. Such a **parking orbit** is convenient for receiving radio signals from one place and transmitting them to another (Fig. 17.4).

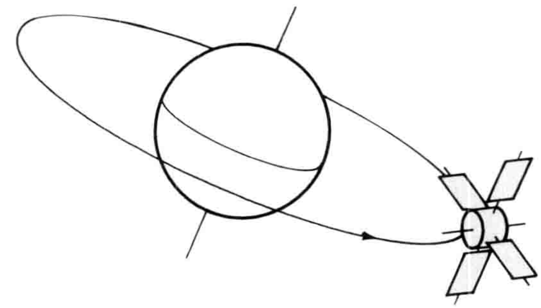


Fig. 17.3 Parking orbit (not to scale)

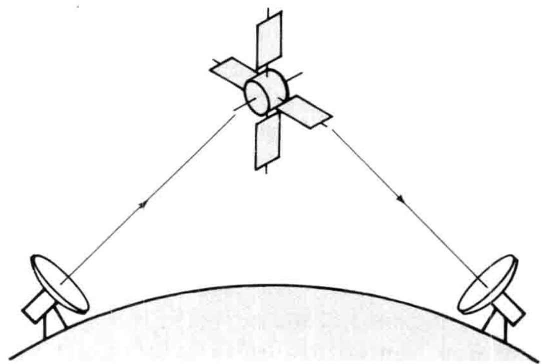


Fig. 17.4 Communication by satellite

The radius of a parking orbit can be found from Eq. (6), using $M = 5.98 \times 10^{24}$ kg and $T = 1$ d $= 8.64 \times 10^4$ s. We find

$$r^3 = \frac{GM}{4\pi^2}T^2$$

$$= (6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}})(5.98 \times 10^{24}\ \mathrm{kg})(8.64 \times 10^4\ \mathrm{s})^2/4\pi^2$$

$$= 75.5 \times 10^{21}\ \mathrm{m^3}$$

$$r = 4.23 \times 10^7\ \mathrm{m} = 42\,300\ \mathrm{km}$$

This is the distance measured from the centre of the earth. The height from the surface of the earth is $r - R \approx 36\,000$ km. Note that r is about 6.6 times the radius of the earth. So the height is very large indeed and is well above the ionosphere which extends from about 80 km to 400 km from the surface of the earth. For this reason short waves in the 6–20 MHz range cannot be used, since they are reflected from the ionosphere. Instead microwaves in the 3–30 GHz range are used for communication through satellites.

At this height the speed v of the satellite is

$$v = \frac{2\pi r}{T} = \frac{2\pi \times 42\,300\text{ km}}{8.64 \times 10^4\text{ s}}$$

$$= 3.08\text{ km s}^{-1}$$

This speed is considerably slower than that of a satellite travelling in a circular orbit close to the earth's surface (7.91 km s^{-1}).

17.3 The earth's gravity

Newton's inverse square law of gravitation refers to two *point* masses a distance r apart. In other words, the size of the objects must be small compared to r. This condition is certainly true for the motion of planets around the sun, or the moon around the earth. We have implicitly assumed that the inverse square law applies even for an object near the earth's surface, i.e. the distance r to the centre of the earth is about the same as the radius R of the earth itself. We have assumed that the whole of the mass of the earth can be regarded as being at its centre. How do we know that this is a valid assumption?

Newton's theoretical calculation

Newton used the following approach to find the gravitational force on a mass m a distance r from the centre of the earth (or any spherically symmetric object). Imagine that the earth is cut into little pieces m_1, m_2, . . . (Fig. 17.5). Each piece can be regarded as a point mass, and its force on the mass m follows the inverse square law. The total force due to all the little pieces is then obtained by vector addition. Newton thus proved that for any *spherical* distribution with total mass M, the total force on a mass m *outside* the distribution is still given by Eq. (4). In other words, all of the mass seems to be concentrated at the earth's centre. If we assume that the earth is a sphere, then this result implies that at any point outside the earth and a distance r from the centre of the earth

$$g' = \frac{F}{m} = G\frac{M}{r^2}$$

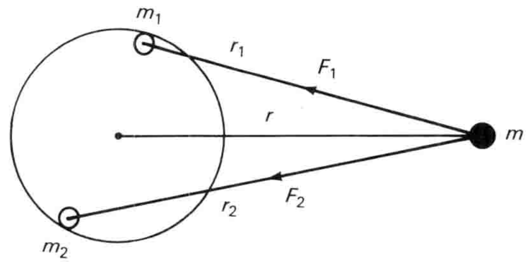


Fig. 17.5

Experimental verification

In this century it has become possible to verify the above conclusion experimentally. Here we give the essential idea. See Section 5.5 for a similar discussion.

Fig. 17.6 shows the flight of the spacecraft Apollo XI from the earth E to the moon, reaching the moon at position $M1$. The spacecraft travelled with the moon from $M1$ to $M2$ and then returned to earth. The figure is drawn according to an observer on earth. Thus the earth appears to have been stationary. (Therefore the observer is non-

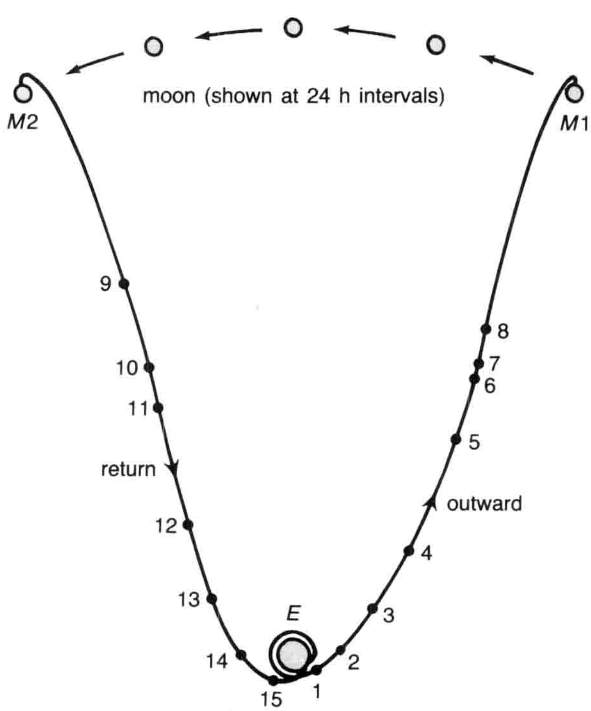


Fig. 17.6 Sketch of typical trajectory for manned moon flight

inertial. However, this effect, being quite small, can be neglected.)

The engine was shut off after point 1 except for a brief interval between 6 and 7, and between 8 and 9. Disregarding these portions, we can say that the spacecraft moved only under the influence of the earth's gravity, the gravitational attraction of the moon being negligible for all the points labelled. The velocity $\vec{v}$ is known for the points 2, 3, ... and also for points 2A, 3A, ... which were a short time interval Δt = 600 s after the respective points 2, 3, ... The data are shown in Table 17.3.

Event	Position	Distance r from centre of Earth/10^6 m	Speed v/m s^{-1}
Rocket not burning: coast begins	1	11.054	8406
	2	26.306	5374
	2A	29.030	5102
	3	54.356	3633
	3A	56.368	3560
	4	95.743	2619
	4A	97.242	2594
	5	169.900	1796
	5A	170.954	1788
No rocket burn until this time	6	209.228	1531.56
3-second burn, changing speed and direction	7	209.232	1527.16
	8	240.624	1356
Coasting	9	241.637	1521
	9A	240.740	1524
	10	209.722	1676
	10A	208.737	1681
	11	170.891	1915
	11A	169.766	1923
	12	96.801	2690
	12A	95.241	2715
	13	56.368	3626
	13A	54.310	3699
	14	28.427	5201
	14A	25.640	5486
	15	13.311	7673
	15A	10.036	8854

Table 17.3

Fig. 17.7a shows the situations at the points 2 and 2A. The change of velocity is the vector difference (Fig. 17.7b)

$$\Delta\vec{v} = \vec{v}' - \vec{v}$$

The force per unit mass is then

$$\frac{\vec{F}}{m} = \vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

which is readily found. Note that the time interval Δt is regarded as sufficiently short, so that $\vec{a}$ can be taken to be the instantaneous acceleration.

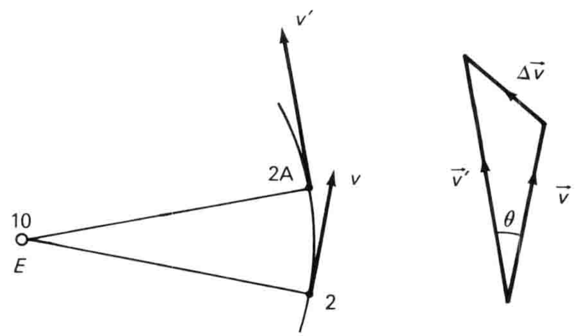


Fig. 17.7 Change of velocity (not to scale)

The calculation is slightly complicated on account of the small angle θ, which is moreover not directly known from Table 17.3. So to illustrate the very simple physics, we imagine a simpler situation — the spacecraft is launched and returns vertically along a straight line (Fig. 17.8). The calculation is shown in *Example 1*.

Fig. 17.8

Example 1

Suppose the spacecraft Apollo XI was launched vertically. The distance r from the centre of the earth and the upward velocity v were recorded for various times (Table 17.3). Calculate the force per unit mass acting on the spacecraft in the interval 2–2A.

Solution

The points 2 and 2A are separated by a time interval $\Delta t = 600\,\text{s}$, during which the average acceleration is

$$a = \frac{\Delta v}{\Delta t} = \frac{(5102 - 5374)\ \text{m s}^{-1}}{600\ \text{s}}$$

$$= -0.453\ \text{m s}^{-2}$$

The time interval is sufficiently short for this to be regarded as the instantaneous acceleration. Since the engine is shut off, there is only the gravitational force F acting on the spacecraft, and $a = F/m$ is just the force per unit mass, i.e. gravitational field strength g'. In magnitude only,

$$g' = \frac{F}{m} = 0.453\,\text{N kg}^{-1}$$

In the same way, we can find the value of g' for various values of r as shown in Table 17.4. A plot of g' versus $1/r^2$ (Fig. 17.9) gives a straight line passing through the origin, showing that the gravitational field strength of the earth obeys the inverse square law.

Points	Gravitational field g'/N kg^{-1}	Average distance r from centre of Earth/10^6 m	$\frac{1}{r^2}/10^{-16}\text{m}^{-2}$
2, 2A	0.453	27.7	13.03
3, 3A	0.122	55.4	3.26
4, 4A	0.042	96.5	1.07
5, 5A	0.013	170.4	0.34

Table 17.4

This result, being consistent with Newton's calculation, can be regarded as a verification of

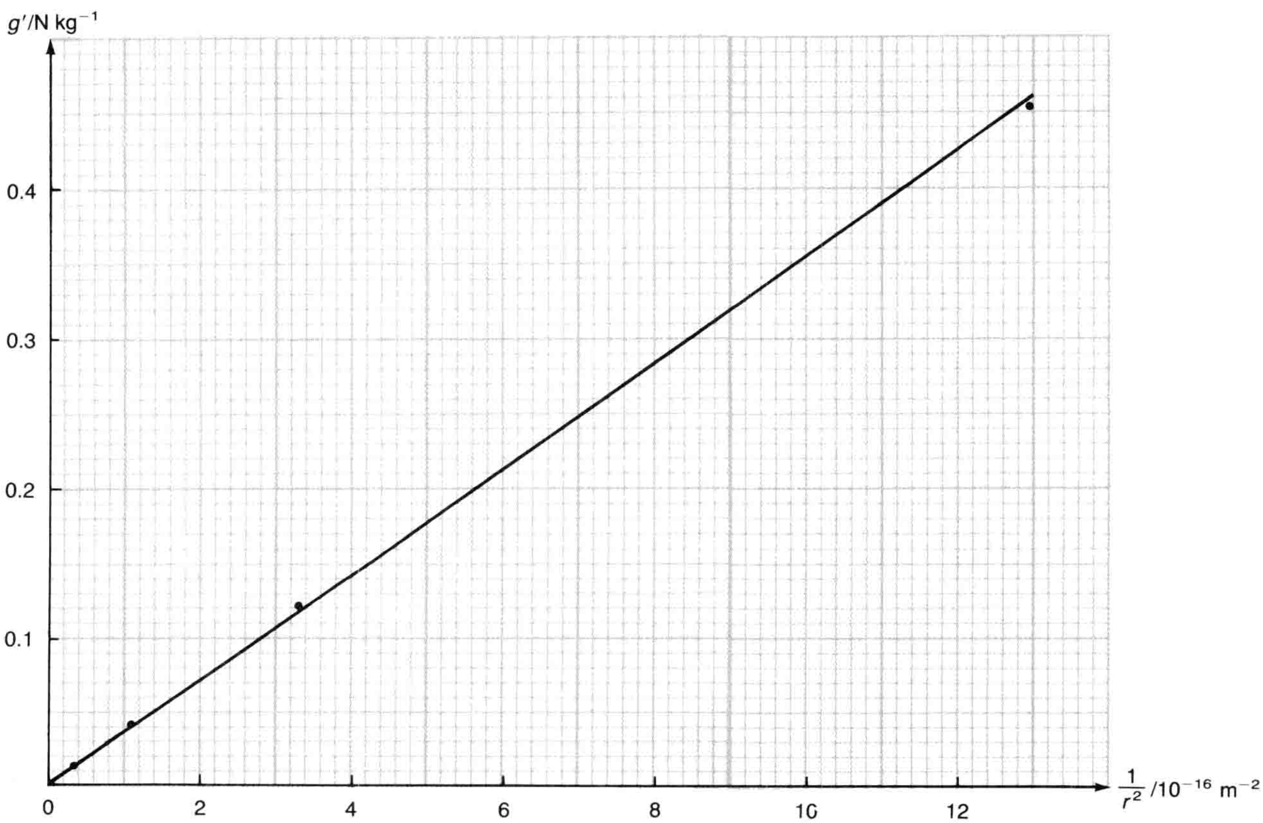


Fig. 17.9 Variation of g' with $1/r^2$

the principle of superposition. We now believe that this principle is valid if the field strength is not too strong, namely if $g' \ll c^2/R$, where c is the velocity of light and R is the typical size of the system. This condition is easily satisfied in the case of the earth.

An approximate form for g' can be derived for points near the earth's surface. At a height h above the ground, i.e. $r = R + h$,

$$g' = G\frac{M}{(R+h)^2} = \frac{GM}{R^2}\left(1 + \frac{h}{R}\right)^{-2}$$
$$\approx \frac{GM}{R^2}\left(1 - 2\frac{h}{R}\right)$$
$$= g\left(1 - \frac{2h}{R}\right) \qquad \text{for } h \ll R$$

where $g = 9.8\,\text{N kg}^{-1}$ is the gravitational field strength at the surface of the earth.

Variation with latitude

The apparent value of g on the earth's surface is not really uniform, but varies with latitude β. The equatorial value g_e ($9.780\,\text{N kg}^{-1}$) is slightly smaller than the polar value g_p ($9.832\,\text{N kg}^{-1}$). Apart from minor local variations, g depends on the latitude in a smooth way, as expressed by the empirical formula

$$g = 9.83221\,(1 - 0.0053\cos^2\beta)\ \text{N kg}^{-1}$$

Most of the variation of g with latitude can be explained by means of the rotation of the earth.

First imagine that the earth does not rotate and is a perfect sphere of radius R. Then g would be the same everywhere on the earth's surface, with the 'standard' value

$$g_0 = \frac{GM}{R^2}$$

Next consider the real earth. Fig. 17.10 shows a mass m resting on the surface of the earth at a latitude β. It is subjected to a gravitational force mg_0 directed towards the centre of the earth, and to the normal reaction force N due to the ground. (If the mass rests on the pan of a spring balance, then N is the 'weight' shown by the balance.) However, these two forces do not cancel each other exactly, because the mass m is not at rest, but performing circular motion about the axis of the earth. The angular velocity is

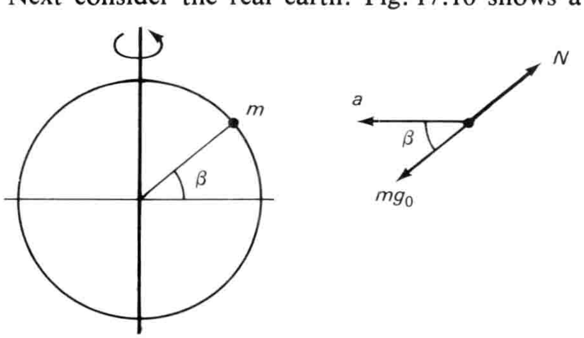


Fig. 17.10

$$\omega = 2\pi\,\text{d}^{-1} = 7.27 \times 10^{-5}\,\text{s}^{-1}$$

while the radius of the circular orbit is $R_\perp = R\cos\beta$. So there is an acceleration

$$a = \omega^2 R_\perp = \omega^2 R\cos\beta$$

which has a radial component given by

$$a_r = a\cos\beta = \omega^2 R\cos^2\beta$$

The component in the tangential direction has a negligible effect.

Now consider Newton's second law, applied to the radial direction

$$mg_0 - N = \text{net force} = ma_r$$
$$N = m(g_0 - a_r) = m(g_0 - \omega^2 R\cos^2\beta)$$

If we express the 'weight' N as mg, then

$$g = g_0 - \omega^2 R\cos^2\beta = g_0(1 - \epsilon\cos^2\beta)$$

where the dimensionless **centrifugal parameter** ϵ is

$$\epsilon = \frac{\omega^2 R}{g_0} = 0.00344$$

If the mass is at the pole, the centripetal acceleration is zero, and the above formula gives $g = g_0$ as expected.

Thus the rotation of the earth gives a correction of the right form, but the coefficient ϵ is about 40% too small. The discrepancy is due to the fact that the earth is oblate, i.e. the polar radius being 0.335% smaller than the equatorial radius. The oblateness has two effects.

(a) Because the earth is not exactly spherical, Eq. (4) is no longer exactly correct. In other words, we cannot imagine the whole mass to be at the centre. This is called the quadrupole moment effect.
(b) Even if Eq. (4) holds exactly true, i.e. $g \propto 1/r^2$, the value of r would depend slightly on latitude.

When the oblateness is taken into account, we arrive at a better theoretical value for the correction factor in the empirical formula. In particular, the coefficient 0.0053 is predicted to within 20%.

The rotation and the oblateness of the earth must not be regarded as entirely independent effects. We believe the earth to be a hot molten liquid when it was first formed. The rotation of the earth caused the 'liquid drop' to assume a slightly oblate shape. Thus even the oblateness effect is indirectly due to the rotation of the earth.

17.4 Field lines

Let us try to represent the gravitational field strength $\vec{g}$ pictorially in the following way.

(a) We draw field lines or **lines of force** (力綫) in the direction of $\vec{g}$.

(b) The density of field lines is proportional to the magnitude of $\vec{g}$.

Field lines never cross; if they did, there would be two directions of $\vec{g}$ at the same point, which is impossible.

Such a picture tells us the magnitude and direction of $\vec{g}$, i.e. everything about $\vec{g}$. Fig. 17.11 shows the field lines near the earth's surface — they are uniform in density, and they point downwards.

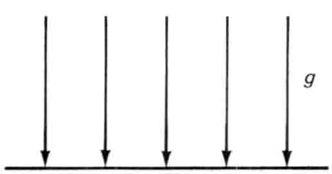


Fig. 17.11

Everything we know about the gravitational field can be re-stated as the properties of field lines. Firstly, suppose a field line forms a loop (Fig. 17.12). Move a mass around the loop in the same direction as g. Then the gravitational force is always doing positive work on the mass. After one cycle, the net work done is non-zero. The force would be non-conservative. But we know that gravity is conservative. (See Chapter 5.) Thus

> Gravitational field lines do not form closed loops.

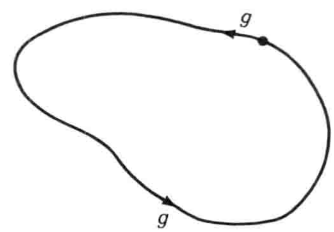


Fig. 17.12

Secondly, consider a mass M at the centre (Fig. 17.13) and compare the field strength at different radii. The field strength, i.e. the density of lines, is proportional to $1/r^2$. However, the lines penetrate a sphere of area $4\pi r^2$. Thus the number of lines penetrating every sphere is the same. In other words, the lines are continuous and penetrate all spheres. In fact they penetrate every closed surface surrounding the mass.

> Gravitational field lines are continuous and only stop on meeting a mass. They do not start or stop in empty space.

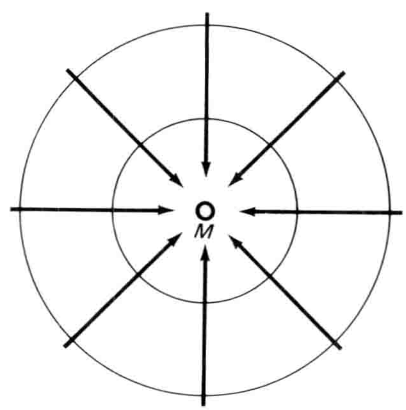


Fig. 17.13

Moreover, on comparing the fields at the same distance (say 1 m) from two masses, it is obvious that the density of lines, i.e. the field strength, should be proportional to each mass. Thus

> The number of lines attached to each object is proportional to the mass of the object.

These rules are equivalent to the inverse square law.

From the above rules we see that the number of lines crossing a given area is an important quantity. So we define the **flux** (通量) Φ through a given area as

$$\Phi = \text{number of lines crossing the area}$$

For convenience, we only draw 1 line crossing a *perpendicular* area of $1\,\text{m}^2$ to represent a field strength of $1\,\text{N}\,\text{kg}^{-1}$. It follows that for an area perpendicular to the field lines (Fig. 17.14)

$$\text{Field strength} = \frac{\Phi}{\text{perpendicular area}} \quad (8)$$

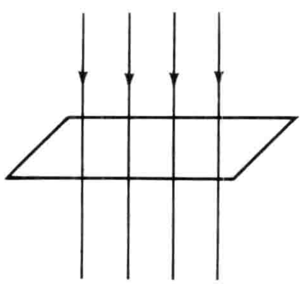


Fig. 17.14 Flux density

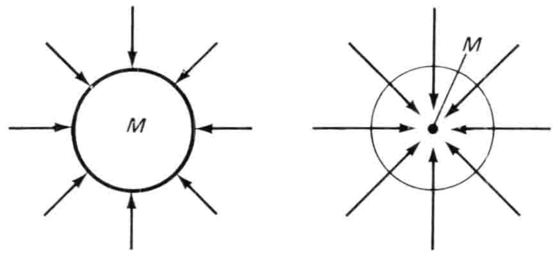
So the field strength g is also called the **flux density** (通量密度). Alternatively we may write

$$\Phi = \text{field strength} \times \text{perpendicular area} \quad (9)$$

As a simple application of these rules, consider a spherical shell of mass M (Fig. 17.15a). There can be no field lines inside the shell, because such lines would have to end in the empty space inside the shell. So we conclude:

> A uniform spherical shell gives rise to zero gravitational field inside the shell.

Now consider the field outside the shell. Compare it with the case of a point mass of the same magnitude at the centre (Fig. 17.15b). The total



(a) (b)

Fig. 17.15

number of lines must be the same in both cases and by symmetry the lines must be evenly spaced and pointing radially inwards. Thus the lines must be exactly the same.

> A uniform spherical shell gives rise to a gravitational field outside the shell which is identical to the field produced by an equivalent point mass at the centre.

Using these results, you can now easily deduce the properties of the gravitational field of the earth.

17.5 Field energy

Potential

We are familiar with the concept of gravitational potential energy near the earth's surface. If we move a mass m up a height h, the work done is mgh. So the PE at the upper position is higher by mgh. If we choose the upper position to be the reference point, then the PE at the lower position is $-mgh$.

If the displacement is not small compared to the radius of the earth, then the force no longer has a constant value mg, and the above derivation needs to be modified. Consider a fixed mass M (such as the earth) and another mass m which can be moved about M. Suppose m is initially at a 'lower position' a distance r from M, and we move it to infinity (Fig. 17.16a). For each little part of the journey,