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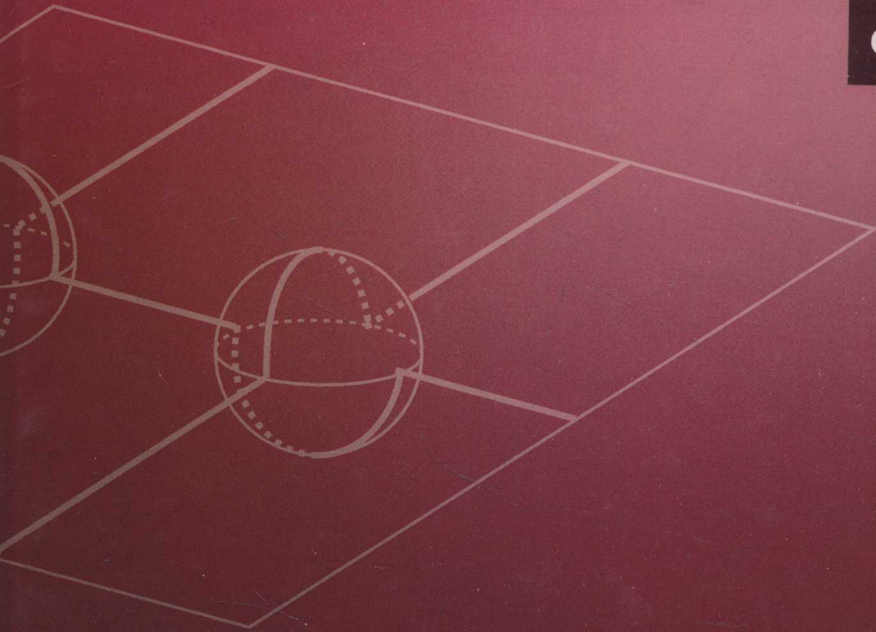

Lectures on Geometry

Edward Witten,
Marc Lackenby,
Martin R. Bridson,
Helmut H. W. Hofer,
Rahul Pandharipande

EDITED BY
N. M. J. Woodhouse

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Lectures on Geometry

Edited by

N. M. J. WOODHOUSE

President, Clay Mathematics Institute

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LECTURES ON GEOMETRY

Series Editor

N. M. J. WOODHOUSE

Preface

This volume contains a collection of papers based on lectures delivered by distinguished mathematicians at Clay Mathematics Institute events over the past few years. It is intended to be the first in an occasional series of volumes of CMI lectures. Although not explicitly linked, the topics in this inaugural volume have a common flavour and a common appeal to all who are interested in recent developments in geometry. They are intended to be accessible to all who work in this general area, regardless of their own particular research interests.

Two Lectures on the Jones Polynomial and Khovanov Homology

Edward Witten

Edward Witten works at the Institute for Advanced Study at Princeton. He is one of the leading figures in contemporary theoretical physics. His chapter is based on two lectures he gave at the Clay Research Conference in 2013. It surveys the groundbreaking work of Witten and his collaborators in fitting Khovanov homology into a quantum field theory framework. In the abstract of his contribution, Witten says: 'I describe a gauge theory approach to understanding quantum knot invariants as Laurent polynomials in a complex variable q. The two main steps are to reinterpret three-dimensional Chern–Simons gauge theory in four-dimensional terms and then to apply electric–magnetic duality. The variable q is associated to instanton number in the dual description in four dimensions.' This hardly does justice to the extraordinary range of ideas and techniques from mathematics and theoretical physics on which his lectures drew in his journey from an elementary starting point in the classical theory of knots. The second lecture was delivered in the Number Theory and Physics workshop at the conference. It takes the story further, describing how Khovanov homology can emerge upon adding a fifth dimension. Along the way, Witten describes many significant new ideas, such as the Kapustin–Witten equations (important in geometric Langlands) and a new approach to evaluating some Feynman integrals via complexification. Witten's approach is very natural, and especially attractive to a geometer, using Picard–Lefschetz theory in an essential way.

Elementary Knot Theory

Marc Lackenby

Marc Lackenby is a Professor of Mathematics at Oxford, with special interests in geometry and topology in three dimensions. His chapter is partly based on a lecture at the Clay Research Conference in 2012. It focuses on identifying some fundamental problems in knot theory that are easy to state but that remain unsolved. A survey of this very active field is given to place these problems into context. Because the tools that are used in knot theory are so diverse, the chapter highlights connections with many other fields of mathematics, including hyperbolic geometry, the theory of computational complexity, geometric group theory (a large area that connects with Bridson's chapter) and Khovanov homology (the subject of Witten's chapter).

Cube Complexes, Subgroups of Mapping Class Groups and Nilpotent Genus

Martin R. Bridson

Martin Bridson is the Whitehead Professor of Mathematics at Oxford, well known for his work in geometric group theory. His contribution is based on the lecture he gave as a Clay Senior Scholar at the Park City Mathematics Institute in 2012. This event is organized each summer by the Institute for Advanced Study at Princeton and is supported by the Clay Mathematics Institute through the appointment of Clay Senior Scholars. The PCMI Scholars provide mathematical leadership for the summer programmes and deliver lectures addressed to a wide mathematical audience. Bridson's chapter focuses on two recent sets of results of his, one on mapping class groups of surfaces and the other on nilpotent genera of groups, both of which illuminate extreme behaviour among finitely presented groups. It provides an extremely useful and readable introduction to an important and lively area.

Polyfolds and Fredholm Theory

Helmut Hofer

Helmut Hofer is a member of the Institute for Advanced Study at Princeton. He has played a major part in the development of symplectic topology. The original version of this important and previously unpublished chapter was written following the Clay Research Conference in 2008, at which Hofer spoke. Since then it has been extended and revised to bring it up to date. The chapter discusses generalized Fredholm theory in polyfolds, an area in which Hofer is a leading figure, with a focus on a particular topic—stable maps—that has a close connection to Gromov–Witten theory. This selection allows Hofer to set his chapter within a broad context. His excellent and full introduction makes accessible the very detailed exposition that follows.

Maps, Sheaves and $K3$ Surfaces

Rahul Pandharipande

Rahul Pandharipande works at ETH Zürich. He is well known for his work with Okounkov, Nekrasov and Maulik on Gromov–Witten theory and Donaldson–Thomas invariants, for which he received a Clay Research Award from CMI in 2013. Pandharipande's chapter also arises from a lecture delivered at the Clay Research Conference in 2008, in which he reviewed his work and that of his collaborators on recent progress in understanding curve counting (Gromov–Witten theory and its cousins) in higher dimensions. Gromov–Witten theory is notoriously hard and is only fully understood in dimensions 0 and 1. Pandharipande describes progress in dimensions 2 and 3. The chapter concisely describes a wide variety of important geometric ideas and useful techniques. It ends by bringing the story up to date with a brief account of the successful proofs of some of the principal conjectures covered in the original lecture.

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Contents

1 Two Lectures on the Jones Polynomial and Khovanov Homology

EDWARD WITTEN

1.1 Lecture One

The Jones polynomial is a celebrated invariant of a knot (or link) in ordinary three-dimensional space, originally discovered by V. F. R. Jones roughly thirty years ago as an offshoot of his work on von Neumann algebras [1]. Many descriptions and generalizations of the Jones polynomial were discovered in the years immediately after Jones's work. They more or less all involved statistical mechanics or two-dimensional mathematical physics in one way or another—for example, Jones's original work involved Temperley–Lieb algebras of statistical mechanics. I do not want to assume that the Jones polynomial is familiar to everyone, so I will explain one of the original definitions.

For brevity, I will describe the "vertex model" (see [2] and also [3], p. 125). One projects a knot to $\mathbb{R}^2$ in such a way that the only singularities are simple crossings and so that the height function has only simple local maxima and minima (Fig. 1.1). One labels the intervals between crossings, maxima and minima by a symbol + or –. One sums over all possible labelings of the knot projection with simple weight functions given in Figs. 1.2 and 1.3. The weights are functions of a variable q. After summing over all possible labelings and weighting each labeling by the product of the weights attached to its crossings, maxima and minima, one arrives at a function of q. The sum turns out to be an invariant of a framed knot.[1] This invariant is a Laurent polynomial in q (times a fixed fractional power of q that depends on the framing). It is known as the Jones polynomial.

Clearly, given the rules stated in the figures, the Jones polynomial for a given knot is completely computable by a finite (but exponentially long) algorithm. The rules, however, seem to have come out of thin air. Topological invariance is not obvious and is proved by checking Reidemeister moves.

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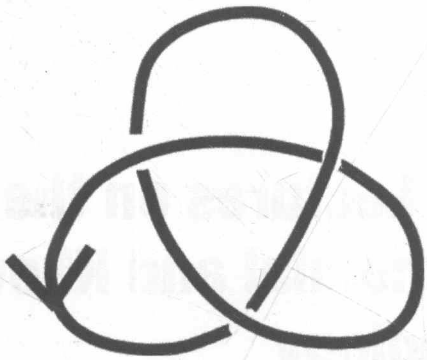

Figure 1.1 A knot in $\mathbb{R}^3$—in this case a trefoil knot—projected to the plane $\mathbb{R}^2$ in a way that gives an immersion with only simple crossings and such that the height function (the vertical coordinate in the figure) has only simple local maxima and minima. In this example, there are three crossings (each of which contributes two crossing points, one on each branch) and two local minima and maxima, making a total of $3 \cdot 2 + 2 + 2 = 10$ exceptional points. Omitting those points divides the knot into 10 pieces that can be labeled by symbols + or −, so the vertex model for this projection expresses the Jones polynomial of the trefoil knot as a sum of 2^{10} terms.

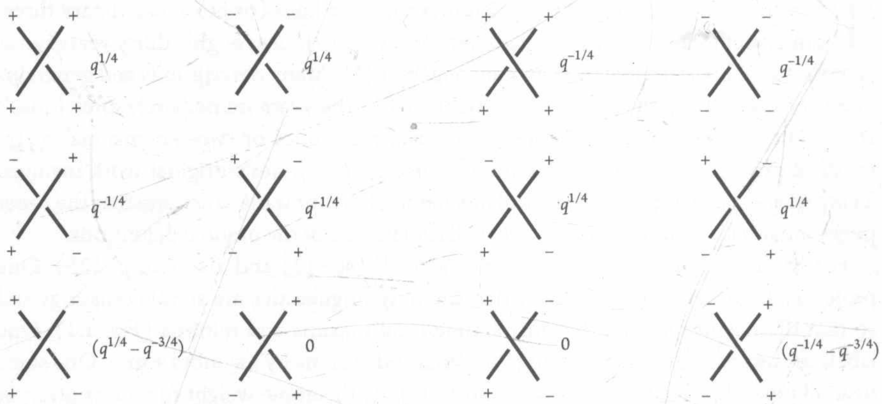

Figure 1.2 The weights of the vertex model for a simple crossing of two strands. (The weights for configurations not shown are 0.)

Other descriptions of the Jones polynomial were found during the same period, often involving mathematical physics. The methods involved statistical mechanics, braid group representations, quantum groups, two-dimensional conformal field theory and more. One notable fact was that conformal field theory can be used [4] to generalize the constructions of Jones to the choice of an arbitrary simple Lie group[2] $G^\vee$ with a labeling of a knot (or of each component of a link) by an irreducible representation $R^\vee$ of $G^\vee$.

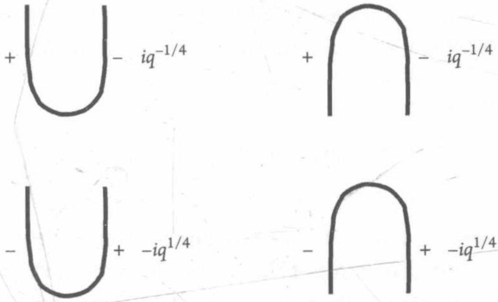

Figure 1.3 The weights of the vertex model for a local maximum or minimum of the height function. (Weights not shown are 0.)

The original Jones polynomial is the case that $G^\vee = \mathrm{SU}(2)$ and $R^\vee$ is the two-dimensional representation.

With these and other clues, it turned out [5] that the Jones polynomial can be described in three-dimensional quantum gauge theory. Here we start with a compact simple gauge group $G^\vee$ (to avoid minor details, we take $G^\vee$ to be connected and simply connected) and a trivial[3] $G^\vee$-bundle $E^\vee \to W$, where W is an oriented three-manifold. Let A be a connection on $E^\vee$. The only gauge-invariant function of A that we can write by integration over W of some local expression, assuming no structure on W except an orientation, is the Chern–Simons function

$$\mathrm{CS}(A) = \frac{1}{4\pi}\int_W \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right). \tag{1.1}$$

Even this function is only gauge-invariant modulo a certain fundamental period. In (1.1), Tr is an invariant and non-degenerate quadratic form on the Lie algebra of $G^\vee$, normalized so that $\mathrm{CS}(A)$ is gauge-invariant mod $2\pi\mathbb{Z}$. For $G^\vee = \mathrm{SU}(n)$ (for some $n \geq 2$), we can take Tr to be the trace in the n-dimensional representation.

The Feynman path integral is now formally an integral over the infinite-dimensional space U of connections:

$$Z_k(W) = \frac{1}{\mathrm{vol}}\int_U DA\ \exp[ik\mathrm{CS}(A)]. \tag{1.2}$$

This is a basic construction in quantum field theory, though unfortunately challenging to understand from a mathematical point of view. Here k has to be an integer since $\mathrm{CS}(A)$ is only gauge-invariant modulo $2\pi\mathbb{Z}$. $Z_k(W)$ is defined with no structure on W except an orientation, so it is an invariant of the oriented three-manifold W. (Here and later, I ignore some details. W actually has to be "framed," as one learns if one follows the logic of "renormalization theory." Also, formally, vol is the volume of the infinite-dimensional group of gauge transformations.)

To include a knot—that is, an embedded oriented circle $K \subset W$—we make use of the *holonomy* of the connection A around W, which we denote by $\mathrm{Hol}(A, K)$. We pick an irreducible representation $R^\vee$ of $G^\vee$ and define

$$\mathcal{W}_{R^\vee}(K) = \mathrm{Tr}_{R^\vee}\, \mathrm{Hol}_K(A) = \mathrm{Tr}_{R^\vee} P \exp\left(-\oint_K A\right), \tag{1.3}$$

where the last expression is the way that physicists often denote the trace of the holonomy. In the context of quantum field theory, the trace of the holonomy is usually called the Wilson loop operator. Then we define a natural invariant of the pair W, K:

$$Z_k(W; K, R^\vee) = \frac{1}{\mathrm{vol}} \int_U DA \exp[ik\mathrm{CS}(A)]\, \mathcal{W}_{R^\vee}(K). \tag{1.4}$$

(Again, framings are needed.)

If we take $G^\vee$ to be $SU(2)$ and $R^\vee$ to be the two-dimensional representation, then $Z_k(W; K, R^\vee)$ turns out to be the Jones polynomial, evaluated at[4]

$$q = \exp\left(\frac{2\pi i}{k+2}\right). \tag{1.5}$$

This statement is justified by making contact with two-dimensional conformal field theory, via the results of [4]. For a particularly direct way to establish the relation to the Knizhnik–Zamolodchikov equations of conformal field theory, see [6]. This relationship between three-dimensional gauge theory and two-dimensional conformal field theory has also been important in condensed matter physics, in studies of the quantum Hall effect and related phenomena.

This approach has more or less the opposite virtues and drawbacks to those of the standard approaches to the Jones polynomial. No projection to a plane is chosen, so topological invariance is obvious (modulo standard quantum field theory machinery), but it is not clear how much one will be able to compute. In other approaches, like the vertex model, there is an explicit finite algorithm for computation, but topological invariance is obscure.

Despite the manifest topological invariance of this approach to the Jones polynomial, there were at least two things that many knot theorists did not like about it. One was simply that the framework of integration over function spaces—though quite familiar to physicists—is difficult to understand mathematically. (A version of this problem is one of the Clay Millennium Problems.) The second is that this method did not give a clear approach to understanding why the usual quantum knot invariants are Laurent polynomials in q. This method, in its original form, gave a definition of the knot invariants only for integer k, and did not explain the existence of an analytic continuation to a function of a complex variable q, let alone the fact that the analytically continued functions are Laurent polynomials. From some points of view, the fact that the invariants are Laurent polynomials is considered sufficiently important that it is part of the name "Jones polynomial." Other approaches to the Jones polynomial—such as the vertex model that we started with—do not obviously give a topological invariant but do obviously give a Laurent polynomial.

Actually, for most three-manifolds, the answer that comes from the gauge theory is the right one. It is special to knots in $\mathbb{R}^3$ that the natural variable is $q = \exp[2\pi i/(k+2)]$ rather than k. The quantum knot invariants on a general three-manifold W are naturally defined only for an integer k and do not have natural analytic continuations to functions of[5] q. This has been the traditional understanding: the gauge theory gives directly a good understanding on a general three-manifold W, but if one wants to understand from three-dimensional gauge theory some of the special things that happen for knots in $\mathbb{R}^3$, one has to begin by relating the gauge theory to one of the other approaches, for instance via conformal field theory.

However, a little over a decade ago, two developments gave clues that there should be another explanation. One of these developments was Khovanov homology, which will be the topic of the second lecture. The other development, which started at roughly the same time, was the "volume conjecture" [7–12]. What I will explain in this lecture started with an attempt to understand the volume conjecture. I should stress that I have not succeeded in finding a quantum field theory explanation for the volume conjecture.[6] However, just understanding a few preliminaries concerning the volume conjecture led to a new point of view on the Jones polynomial. This is what I aim to explain. Since this is the case, I will actually not give a precise statement of the volume conjecture.

To orient ourselves, let us just ask how the basic integral

$$Z_k(W) = \frac{1}{\mathrm{vol}} \int_U DA \, \exp[ik\mathrm{CS}(A)] \tag{1.6}$$

behaves for large k. It is an infinite-dimensional analog of a finite-dimensional oscillatory integral such as the one that defines the Airy function

$$F(k; t) = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp[ik(x^3 + tx)], \tag{1.7}$$

where we assume that k and t are real. Taking $k \to \infty$ with fixed t, the integral vanishes exponentially fast if there are no real critical points ($t > 0$) and is a sum of oscillatory contributions of real critical points if there are any ($t < 0$). The same logic applies to the infinite-dimensional integral for $Z_k(W)$. The critical points of $\mathrm{CS}(A)$ are flat connections, corresponding to homomorphisms $\rho : \pi_1(W) \to G$, so the asymptotic behavior of $Z_k(W)$ for large k is given by a sum of oscillatory contributions associated to such homomorphisms. (This has been shown explicitly in examples [14, 15].)

The volume conjecture arises if we specialize to knots in $\mathbb{R}^3$, so that—as one knows from any approach to the Jones polynomial other than that via Chern–Simons gauge theory—k does not have to be an integer. Usually the case $G^\vee = SU(2)$ is assumed and we let $R^\vee$ be the n-dimensional representation of $SU(2)$. The corresponding knot invariant is called the colored Jones polynomial. We take $k \to \infty$ through non-integer values, with fixed k/n. A choice that is sufficient to illustrate the main points is to set $k = k_0 + n$, where k_0 is a fixed complex number and we take $n \to \infty$ (through integer values). The behavior of the colored Jones polynomial in this limit has been studied for a variety of knots, using approaches to the knot invariants in which there is no restriction to integer k, for example the approach via quantum groups. Very interesting results have emerged from this work [7–12]. Trying to understand these results via path integrals was the motivation for what I am describing in this lecture.

What emerged from study of the limit $n \to \infty$ with $k = k_0 + n$ is very suggestive of Chern–Simons gauge theory, but with a crucial twist. In examples that have been studied, the large-n behavior can be interpreted in terms of a sum of critical points of the Chern–Simons path integral, but now these are *complex* critical points. By a complex critical point, I mean simply a critical point of the analytic continuation of the function $\mathrm{CS}(A)$.

We make this analytic continuation simply by replacing the Lie group $G^\vee$ with its complexification $G^\vee_{\mathbb{C}}$, replacing the $G^\vee$-bundle $E^\vee \to W$ with its complexification, which is a $G^\vee_{\mathbb{C}}$ bundle $E^\vee_{\mathbb{C}} \to W$, and replacing the connection A on $E^\vee$ by a connection $\mathcal{A}$ on $E^\vee_{\mathbb{C}}$, which we can think of as a complex-valued connection. Once we do this, the function $\mathrm{CS}(A)$ on the space U of connections on $E^\vee$ can be analytically continued to a holomorphic function $\mathrm{CS}(\mathcal{A})$ on $\mathcal{U}$, the space of connections on $E^\vee_{\mathbb{C}}$. This function is defined by the "same formula" with A replaced by $\mathcal{A}$:

$$\mathrm{CS}(\mathcal{A}) = \frac{1}{4\pi}\int_W \mathrm{Tr}\left(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right). \tag{1.8}$$

On a general three-manifold W, a critical point of $\mathrm{CS}(\mathcal{A})$ is simply a complex-valued flat connection, corresponding to a homomorphism $\rho : \pi_1(W) \to G^\vee_{\mathbb{C}}$.

In the case of the volume conjecture with $W = \mathbb{R}^3$, the fundamental group is trivial, but we are supposed to also include a holonomy or Wilson loop operator $\mathcal{W}_{R^\vee}(K) = \mathrm{Tr}_{R^\vee}\,\mathrm{Hol}_K(A)$, where $R^\vee$ is the n-dimensional representation of $\mathrm{SU}(2)$. When we take $k \to \infty$ with fixed k/n, this holonomy factor affects what we should mean by a critical point.[7] A full explanation would take us too far afield, and instead I will just give the answer: the right notion of a complex critical point for the colored Jones polynomial is a homomorphism $\rho : \pi_1(W\backslash K) \to G^\vee_{\mathbb{C}}$, with a monodromy around K whose conjugacy class is determined by the ratio n/k. What is found in work on the "volume conjecture" is that (in examples that have been studied) the colored Jones polynomial for $k \to \infty$ with fixed n/k is determined by such a complex critical point.

Physicists know about various situations (involving "tunneling" problems) in which a path integral is dominated by a complex critical point, but usually this is a complex critical point that makes an exponentially small contribution. There is a simple reason for this. Usually in quantum mechanics, one is computing a probability amplitude. Since probabilities cannot be bigger than 1, the contribution of a complex critical point to a probability amplitude can be exponentially small but it cannot be exponentially large. What really surprised me about the volume conjecture is that, for many knots (knots with hyperbolic complement in particular), the dominant critical point makes an exponentially *large* contribution. In other words, the colored Jones polynomial is a sum of oscillatory terms for $n \to \infty$, $k = k_0 + n$ if k_0 is an integer, but it grows exponentially in this limit as soon as k_0

is not an integer. (Concretely, this is because $k\mathrm{CS}(\mathcal{A})$ evaluated at the appropriate critical point has a negative imaginary part, so $\exp[ik\mathrm{CS}(\mathcal{A})]$ grows exponentially for large k.)

There is no contradiction with the statement that quantum mechanical probability amplitudes cannot be exponentially large, because as soon as k_0 is not an integer, we are no longer studying a physically sensible quantum mechanical system. But it seemed puzzling that making k_0 non-integral, even if still real, can change the large-n behavior so markedly. However, it turns out that a simple one-dimensional integral can do the same thing:

$$I(k,n) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{ik\theta}\, e^{2in\sin\theta}. \tag{1.9}$$

We want to think of k and n as analogs of the integer-valued parameters in Chern–Simons gauge theory that we call by the same names. (In our model problem, k is naturally an integer, but there is no good reason for n to be an integer. So the analogy is not perfect.) If one takes k, n to infinity with a fixed (real) ratio and maintaining the integrality of k, then the integral $I(k,n)$ has an oscillatory behavior, dominated by the critical points of the exponent $f = k\theta + 2n\sin\theta$, if k/n is such that there are critical points for real θ. Otherwise, the integral vanishes exponentially fast for large k.

Now, to imitate the situation considered in the volume conjecture, we want to analytically continue away from integer values of k. The integral $I(k,n)$ obeys Bessel's equation (as a function of n) for any integer k. We want to think of Bessel's equation as the analog of the "Ward identities" of quantum field theory, so in the analytic continuation of $I(k,n)$ away from integer k, we want to preserve Bessel's equation. The proof of Bessel's equation involves integration by parts, so it is important that we are integrating all the way around the circle and that the integrand is continuous and single-valued on the circle. That is why k has to be an integer.

The analytic continuation of $I(k,n)$, preserving Bessel's equation, was known in the nineteenth century. We first set $z = e^{i\theta}$, so our integral becomes

$$I(k,n) = \oint \frac{\mathrm{d}z}{2\pi i} z^{k-1} \exp[n(z - z^{-1})]. \tag{1.10}$$

Here the integral is over the unit circle in the z-plane. At this point, k is still an integer. We want to get away from integer values while still satisfying Bessel's equation. If $\mathrm{Re}\, n > 0$, this can be done by switching to the integration cycle shown in Fig. 1.4.

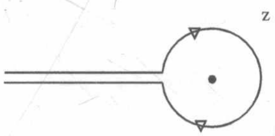

Figure 1.4 The contour used in analytic continuation of the solution of Bessel's equation.

The integral on the new cycle converges (if $\mathrm{Re}\, n > 0$), and it agrees with the original integral on the circle if k is an integer, since the extra parts of the cycle cancel. But the new cycle gives a continuation away from integer k, still obeying Bessel's equation. There is no difficulty in the integration by parts used to prove Bessel's equation, since the integral on the chosen cycle is rapidly convergent at infinity.

How does the integral on the new cycle behave in the limit $k, n \to \infty$ with fixed k/n? If k is an integer and n is real, then the integral is oscillatory or exponentially damped, as I have stated before, depending on the ratio k/n. But as soon as k is not an integer (even if k and n remain real), the large-k behavior with fixed k/n is one of exponential growth, for a certain range of k/n, rather as is found for the colored Jones polynomial. Unfortunately, even though it is elementary, a full explanation of this statement would involve a bit of a digression. (Details can be found, for example, in [13], Section 3.5.) Here I will just explain the technique that one can use to make this analysis, since this will show the technique that we will follow in taking a new look at the Jones polynomial.

We are trying to do an integral of the generic form

$$\int_\Gamma \frac{dz}{2\pi iz} \exp[kF(z)], \tag{1.11}$$

where $F(z)$ is a holomorphic function and Γ is a cycle, possibly not compact, on which the integral converges. In our case,

$$F(z) = \log z + \lambda(z - z^{-1}), \qquad \lambda = n/k. \tag{1.12}$$

We note that because of the logarithm, $F(z)$ is multivalued. To do the analysis properly, we should work on a cover of the punctured z-plane parametrized by $w = \log z$ on which F is single-valued:

$$F(w) = w + \lambda(e^w - e^{-w}). \tag{1.13}$$

The next step is to find a useful description of all possible cycles on which the desired integral, which now is

$$\int_\Gamma \frac{dw}{2\pi i} \exp[kF(w)], \tag{1.14}$$

converges.

Morse theory gives an answer to this question. We consider the function $h(w, \overline{w}) = \mathrm{Re}[kF(w)]$ as a Morse function. Its critical points are simply the critical points of the holomorphic function F, and so in our example they obey

$$1 + \lambda(e^w + e^{-w}) = 0. \tag{1.15}$$

The key step is now the following. To every critical point p of F, we can define an integration cycle Γ_p, called a Lefschetz thimble, on which the integral we are trying to do