Applied Mathematical Sciences 127

Victor Isakov

Inverse Problems for Partial Differential Equations

Second Edition

偏微分方程中的逆问题 第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

Victor Isakov

Inverse Problems for Partial Differential Equations

Second Edition

Springer

图书在版编目（CIP）数据

偏微分方程中的逆问题：第2版 = Inverse problems for partial differential equations 2nd ed.：英文/（美）艾赛科威（Isakov，V.）著．—影印本．—北京：世界图书出版公司北京公司，2013.10

ISBN 978-7-5100-7023-5

Ⅰ．①偏…　Ⅱ．①艾…　Ⅲ．①偏微分方程—逆问题—研究—英文　Ⅳ．①O175

中国版本图书馆CIP数据核字（2013）第249240号

书　　名：	Inverse Problems for Partial Differential Equations 2nd ed.
作　　者：	Victor Isakov
中 译 名：	偏微分方程中的逆问题 第2版
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河市国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街137号100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@wpcbj.com.cn
开　　本：	24开
印　　张：	15.5
版　　次：	2014年3月
版权登记：	图字：01-2013-2914
书　　号：	978-7-5100-7023-5　　定　　价：59.00元

To my wife Julie

Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them a result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backward, or analytically.

—Arthur Conan Doyle, *A Study in Scarlet*

Preface to the Second Edition

In 8 years after publication of the first version of this book, the rapidly progressing field of inverse problems witnessed changes and new developments. Parts of the book were used at several universities, and many colleagues and students as well as myself observed several misprints and imprecisions. Some of the research problems from the first edition have been solved. This edition serves the purposes of reflecting these changes and making appropiate corrections. I hope that these additions and corrections resulted in not too many new errors and misprints.

Chapters 1 and 2 contain only 2–3 pages of new material like in sections 1.5, 2.5. Chapter 3 is considerably expanded. In particular we give more convenient definition of pseudo-convexity for second order equations and included boundary terms in Carleman estimates (Theorem 3.2.1′) and Counterexample 3.2.6. We give a new, shorter proof of Theorem 3.3.1 and new Theorems 3.3.7, 3.3.12, and Counterexample 3.3.9. We revised section 3.4, where a new short proof of exact observability inequality in given: proof of Theorem 3.4.1 and Theorems 3.4.3, 3.4.4, 3.4.8, 3.4.9 are new. Section 3.5 is new and it exposes recent progress on Carleman estimates, uniqueness and stability of the continuation for systems. In Chapter 4 we added to sections 4.5, 4.6 some new material on size evaluation of inclusions and on small inclusions. Chapter 5 contains new results on identification of an elliptic equation from many local boundary measurements (Theorem 5.2.2′, Lemma 5.3.8), a counterexample to stability, a brief description of recent complete results on uniqueness of conductivity in the plane case, some new results on identification of many coefficients and of quasilinear equations insectiosn 5.5, 5.6, and changes and most recent results on uniqueness for some important systems, like isotropic elasticity systems. In Chapter 7 we inform about new developments in boundary rigidity problem. Section 7.4 now exposes a complete solution of the uniqueness problem in the attenuated plane tomography over straight lines (Theorem 7.4.1) and an outline of relevant new methods and ideas. In section 8.2 we give a new general scheme of obtaining uniqueness results based on Carleman estimates and applicable to a wide class of partial differential equations and systems (Theorem 8.2.2) and describe recent progress on uniqueness problem for linear isotropic elasticity system. In Chapter 9 we expanded the exposition in section 9.1

to reflect increasing importance of the final overdetermination (Theorems 9.1.1, 9.1.2) . In section 9.2 we expose new stability estimate for the heat equation transform (Theorem 9.2.1' Lemma 9.2.2). New section 9.3 is dedicated to emerging financial applications: the inverse option pricing problem. We give more detailed proofs in section 9.5 (Lemma 9.5.5 and proof of Theorem 9.5.2). In Chapter 10 we added a brief description of a new efficient single layer algorithm for an imporatnt inverse problem in acoustics in section 10.2 and a new section 10.5 on so-called range tests for numerical solutions of overdermined inverse problems.

Many exercises have been solved by students, while most of the research problems await solutions. Chapter 7 of the final version of the manuscript have been read by Alexander Bukhgeim, who found several misprints and suggested many corrections. The author is grateful to him for attention and help. He also thanks the National Science Foundation for long-term support of his research, which stimulated his research and the writing of this revision.

Wichita, Kansas Victor Isakov

Preface to the First Edition

This book describes the contemporary state of the theory and some numerical aspects of inverse problems in partial differential equations. The topic is of substantial and growing interest for many scientists and engineers, and accordingly to graduate students in these areas. Mathematically, these problems are relatively new and quite challenging due to the lack of conventional stability and to nonlinearity and nonconvexity. Applications include recovery of inclusions from anomalies of their gravitational fields; reconstruction of the interior of the human body from exterior electrical, ultrasonic, and magnetic measurements, recovery of interior structural parameters of detail of machines and of the underground from similar data (non-destructive evaluation); and locating flying or navigated objects from their acoustic or electromagnetic fields. Currently, there are hundreds of publications containing new and interesting results. A purpose of the book is to collect and present many of them in a readable and informative form. Rigorous proofs are presented whenever they are relatively short and can be demonstrated by quite general mathematical techniques. Also, we prefer to present results that from our point of view contain fresh and promising ideas. In some cases there is no complete mathematical theory, so we give only available results. We do not assume that a reader possesses an enormous mathematical technique. In fact, a moderate knowledge of partial differential equations, of the Fourier transform, and of basic functional analysis will suffice. However, some details of proofs need quite special and sophisticated methods, but we hope that even without completely understanding these details a reader will find considerable useful and stimulating material. Moreover, we start many chapters with general information about the direct problem, where we collect, in the form of theorems, known (but not simple and not always easy to find) results that are needed in the treatment of inverse problems. We hope that this book (or at least most of it) can be used as a graduate text. Not only do we present recent achievements, but we formulate basic inverse problems, discuss regularization, give a short review of uniqueness in the Cauchy problem, and include several exercises that sometimes substantially complement the book. All of them can be solved by using some modification of the presented methods.

Parts of the book in a preliminary form have been presented as graduate courses at the Johannes-Kepler University of Linz, at the University of Kyoto, and at Wichita State University. Many exercises have been solved by students, while most of the research problems await solutions. Parts of the final version of the manuscript have been read by Ilya Bushuyev, Alan Elcrat, Matthias Eller, and Peter Kuchment, who found several misprints and suggested many corrections. The author is grateful to these colleagues for their attention and help. He also thanks the National Science Foundation for long-term support of his research, which stimulated the writing of this book.

Wichita, Kansas Victor Isakov

Contents

1

Inverse Problems

In this chapter we formulate basic inverse problems and indicate their applications. The choice of these problems is not random. We think that it represents their interconnections and some hierarchy.

An inverse problem assumes a direct problem that is a well-posed problem of mathematical physics. In other words, if we know completely a "physical device," we have a classical mathematical description of this device including uniqueness, stability, and existence of a solution of the corresponding mathematical problem. But if one of the (functional) parameters describing this device is to be found (from additional boundary/experimental) data, then we arrive at an inverse problem.

1.1 The inverse problem of gravimetry

The gravitational field u, which can be measured and perceived by the gravitational force ∇u and which is generated by the mass distribution f, is a solution to the Poisson equation

$$-\Delta u = f \tag{1.1.1}$$

in $\mathbb{R}^3$, where $\lim u(x) = 0$ as $|x|$ goes to $+\infty$. For modeling and for computational reasons, it is useful to consider as well the plane case ($\mathbb{R}^2$). Then the behavior at infinity must be $u(x) = Cln|x| + u_0(x)$, where u_0 goes to zero at infinity. One assumes that f is zero outside a bounded domain Ω, which is a ball or a body close to a ball (earth) in gravimetry. The direct problem of gravimetry is to find u given f. This is a well-posed problem: Its solution exists for any integrable f, and even for any distribution that is zero outside Ω; it is unique and stable with respect to standard functional spaces. As a result, the boundary value problem (1.1.1) can be solved numerically by using difference schemes, although these computations are not very easy in the three-dimensional case. This solution is given by the Newtonian potential

$$u(x) = \int_{\Omega} k(x-y)f(y)dy, \qquad k(x) = 1/(4\pi|x|) \tag{1.1.2}$$

(or $k(x) = -1/(2\pi)\ln|x|$ in $\mathbb{R}^2$). Practically we perceive and can measure only the gravitational force ∇u.

The *inverse problem of gravimetry* is to find f given ∇u on Γ, which is a part of the boundary $\partial\Omega$ of Ω.

This problem was actually formulated by Laplace, but the first (and simplest) results were obtained only by Stokes in the 1860s and Herglotz about 1910 [Her]. We will analyze this problem in Sections 2.1–2.2 and 4.1. There is an advanced mathematical theory of this problem presented in a book of the author [Is4]. It is fundamental in geophysics, since it simulates recovery of the interior of the earth from boundary measurements of the gravitational field. Unfortunately, there is a strong nonuniqueness of f for a given gravitational potential outside Ω. However, if we look for a more special type of f (like harmonic functions, functions independent of one variable, or characteristic functions $\chi(D)$ of unknown domains D inside Ω), then there is uniqueness, and f can be recovered from u given outside Ω, theoretically and numerically. In particular, one can show uniqueness of $f = \chi(D)$ when D is either star-shaped with respect to its center of gravity or convex with respect to one of the coordinates.

An important feature of the inverse problem of gravimetry is its ill-posedness, which creates many mathematical difficulties (absence of existence theorems due to the fact that ranges of operators of this problem are not closed in classical functional spaces) and numerical difficulties (stability under constraints is (logarithmically) weak, and therefore convergence of iterative algorithms is very slow, so numerical errors accumulate and do not allow good resolution). In fact, it was Tikhonov who in 1944 observed that introduction of constraints can restore some stability to this problem, and this observation was one of starting points of the contemporary theory of ill-posed problems.

This problem is fundamental in recovering the density of the earth by interpreting results of measurements of the gravitation al field (gravitational anomalies). Another interesting application is in gravitational navigation. One can measure the gravitational field (from satellites) with quite high precision, then possibly find the function f that produces this field, and use these results to navigate aircrafts. To navigate aircraft one needs to know u near the surface of the earth Ω, and finding f supported in $\bar{\Omega}$ gives u everywhere outside of Ω by solving a much easier direct problem of gravimetry. The advantage of this method is that the gravitational field is the most stationary and stable of all known physical fields, so it is most suitable for navigation. The inverse problem here is used to record and store information about the gravitational field. This problem is quite unstable, but still manegeable. We discuss this problem in Sections 2.2, 2.3, 3.3, 4.1, and in Chapter 10.

Inverse gravimetry is a classical example of an inverse source problem, where one is looking for the right side of a differential equation (or a system of equations) from extra boundary data. Let us consider a simple example: in the second-order ordinary differential equation $-u'' = f$ on $\Omega = (-1, 1)$ in $\mathbb{R}$. Let $u_0 = u(-1)$, $u_1 = u'(-1)$; then

$$u(x) = u_0 + u_1(x+1) - \int_{-1}^{x} (x-y)f(y)dy \text{ when } -1 < x < 1.$$

Prescribing the Cauchy data u, u' at $t = 1$ is equivalent to the prescription of two integrals

$$\int_\Omega (1-y)f(y)dy \text{ and } \int_\Omega f(y)dy.$$

We cannot determine more given the Cauchy data at $t = -1, 1$, no matter what is the original Cauchy data. The same information about f is obtained if we prescribe any u on $\partial\Omega$ and if in addition we know u' on $\partial\Omega$. In particular, nonuniqueness is substantial: one cannot find a function from two numbers. If we add to f any function f_0 such that

$$\int_\Omega v(y)f_0(y)dy = 0$$

for any linear function v (i.e., for any solution of the adjoint equation $-v'' = 0$), then according to the above formulae we will not change the Cauchy data on $\partial\Omega$. The situation with partial differential equations is quite similar, although more complicated.

If ∇u is given on Γ, then u can be found uniquely outside Ω by uniqueness in the Cauchy problem for harmonic functions using the assumptions on the behavior at infinity. Observe that given u on $\partial\Omega \subset \mathbb{R}^3$ one can solve the exterior Dirichlet problem for u outside Ω and find $\partial_\nu u$ on $\partial\Omega \in Lip$, so in fact we are given the Cauchy data there.

Exercise 1.1.1. Assume that Ω is the unit disk $\{|x| < 1\}$ in $\mathbb{R}^2$.

Show that a solution $f \in L_\infty(\Omega)$ of the inverse gravimetry problem that satisfies one of the following three conditions is unique. (1) It does not depend on $r = |x|$. (2) It satisfies the second-order equation $\partial_2^2 f = 0$. (3) It satisfies the Laplace equation $\Delta f = 0$ in Ω.

In fact, in the cases (2) and (3), Ω can be any bounded domain with $\partial\Omega \in C^3$ with connected $\mathbb{R}^2\backslash\bar{\Omega}$. {*Hint*: to handle case (1) consider $v = r\partial_r u - 2u$ and observe that v is harmonic in Ω. Determine v in Ω by solving the Dirichlet problem and then find f. In cases (2) and (3) introduce new unknown (harmonic in Ω) functions $v = \partial_2^2 u$ and $v = \Delta u$.}

Exercise 1.1.2. In the situation of Exercise 1.1.1 prove that a density $f(r)$ creates zero exterior potential if and only if

$$\int_0^1 rf(r)dr = 0.$$

{*Hint*: make use of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and of the expression for the Laplacian in polar coordinates,

$$\Delta = r^{-1}(\partial_r(r\partial_r) + \partial_\theta(r^{-1}\partial_\theta)).$$

Observe that for such f the potential u does not depend on θ, and perform an analysis similar to that given above for the simplest differential equation of second order.}

What we discussed briefly above can be called the density problem. It is linear with respect to f. The domain problem when one is looking for the unknown D is apparently nonlinear and seems (and indeed is) more difficult. In this introduction we simply illustrate it by recalling that the Newtonian potential U of the ball $D = B(a; R) \subset \mathbb{R}^3$ of constant density ρ is given by the formulae

$$(1.1.3) \quad U(x; \rho\chi(B(a; R))) = \begin{bmatrix} R^3\rho/3|x-a|^{-1} & \text{when } |x-a| \geq R; \\ R^2\rho/2 - \rho/6|x-a|^2 & \text{when } |x-a| < R. \end{bmatrix}$$

These formulae imply that a ball and its constant density cannot be simultaneously determined by their exterior potential ($|x - a| > R$). One can only find $R^3\rho$. Moreover, according to (1.1.2) and (1.1.3), the exterior Newtonian potential of the annulus $A(a; R_1, R_2) = B(a; R_2)\backslash \bar{B}(a; R_1)$ is $(R_2^3 - R_1^3)\rho/3|x-a|^2$, so only $\rho(R_2^3 - R_1^3)$ can be found. In fact, in this example the cavity of an annulus further deteriorates uniqueness. The formulae (1.1.3) can be obtained by observing the rotational (around a) invariancy of the equation (1.1.1) when $f = \rho\chi(B(a; R))$ and using this equation in polar coordinates together with the continuity of the potential and first order derivatives of the potential at ∂D.

We will give more detail on interesting and not completely resolved inverse problem of gravimetry in Section 4.1, observing that starting from the pioneering work of P. Novikov [No], uniqueness and stability results have been obtained by Prilepko [Pr], [PrOV], Sretensky, Tikhonov, and the author [Is4].

There is another interesting problem of potential theory in geophysics, that of finding the shape of the geoid D given the gravitational potential at its surface. Mathematically, like the domain problem in gravimetry, it is a free boundary problem that consists in finding a bounded domain D and a function u satisfying the conditions

$$\Delta u = \rho \text{ in } D \subset \mathbb{R}^3, \qquad \Delta u = 0 \text{ outside } \bar{D},$$
$$u, \nabla u \in C(\mathbb{R}^3), \quad \lim u(x) = 0 \text{ as } |x| \to \infty,$$
$$u = g_0 \text{ on } \partial D,$$

where g_0 is a given function. To specify the boundary condition, we assume that D is star-shaped, so it is given in polar coordinates (r, σ) by the equation $r < d(\sigma)$, $|\sigma| = 1$. Then the boundary condition should be understood as $u(d(\sigma)\sigma) = g_0(\sigma)$, where g_0 is a given function on the unit sphere. This problem is called the Molodensky problem, and it was the subject of recent intensive study by both mathematicians and geophysicists. Again, despite certain progress, there are many challenging questions, in particular, the global uniqueness of a solution is not known.

To describe electrical and magnetic phenomena one makes use of single- and double-layer potentials

$$U^{(1)}(x; gd\Gamma) = \int_\Gamma K(x,y)g(y)d\Gamma(y)$$

and

$$U^{(2)}(x; gd\Gamma) = \int_\Gamma \partial_{\nu(y)} K(x, y) g(y) d\Gamma(y)$$

distributed with (measurable and bounded) density g over a piecewise-Lipschitz bounded surface Γ in $\mathbb{R}^3$. As in inverse gravimetry, one is looking for g and Γ (or for one of them) given one of these potentials outside a reference domain Ω. The inverse problem for the single-layer potential can be used, for example, in gravitational navigation: it is probably more efficient to look for a single layer distribution g instead of the volume distribution f. As a good example of a practically important problem about double layer potentials we mention that of exploring the human brain to find active parts of its surface Γ_c (cortical surface). The area of active parts occupy not more than 0.1 of area of Γ_c. They produce a magnetic field that can be described as the double-layer potential distributed over Γ_c with density $g(y)$, and one can (quite precisely) measure this field outside the head Ω of the patient. We have the integral equation of the first kind

$$G(x) = \int_{\Gamma_c} \partial_{\nu(y)} K(x, y) g(y) d\Gamma(y), \quad x \in \partial\Omega,$$

where Γ_c is a given C^1-surface, $\bar{\Gamma}_c \subset \Omega$, and $g \in L_\infty(\Gamma_c)$ is an unknown function. In addition to its obvious ill-posedness, an intrinsic feature of this problem is the complicated shape of Γ_c. There have been only preliminary attempts to solve it numerically. No doubt a rigorous mathematical analysis of the problem (asymptotic formulae for the double-layer potential when Γ_c is replaced by a closed smooth surface or, say, use of homogenization) could help a lot.

In fact, it is not very difficult to prove uniqueness of g (up to a constant) with the given exterior potential of the double layer.

We observe that in inverse source problems one is looking for a function f entering the partial differential equation $-\Delta u = f$ when its solution u is known outside Ω. If one allows f to be a measure or a distribution of first order, then the inverse problems about the density g of a single or double layer can be considered as an inverse source problem with $f = d\Gamma$ or $f = g\partial_\nu(d\Gamma)$.

1.2 The inverse conductivity problem

The conductivity equation for electric (voltage) potential u is

(1.2.1) $$\operatorname{div}(a\nabla u) = 0 \text{ in } \Omega.$$

For a unique determination of u one can prescribe at the boundary the Dirichlet data

(1.2.2) $$u = g_0 \text{ on } \partial\Omega.$$

Here we assume that a is a scalar function, $0 < \varepsilon_0 \le a$, that is measurable and bounded. In this case one can show that there is a unique solution $u \in H_{(1)}(\Omega)$ to the