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Vladimir I. Nekorkin

Introduction to Nonlinear Oscillations

非线性振动理论导引

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With 137 figures

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NONLINEAR PHYSICAL SCIENCE

非线性物理科学

NONLINEAR PHYSICAL SCIENCE

Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

Topics of interest in *Nonlinear Physical Science* include but are not limited to:

- New findings and discoveries in nonlinear physics and mathematics
- Nonlinearity, complexity and mathematical structures in nonlinear physics
- Nonlinear phenomena and observations in nature and engineering
- Computational methods and theories in complex systems
- Lie group analysis, new theories and principles in mathematical modeling
- Stability, bifurcation, chaos and fractals in physical science and engineering
- Nonlinear chemical and biological physics
- Discontinuity, synchronization and natural complexity in the physical sciences

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Preface

At the foundation of this course material are lectures on a general course in the theory of oscillations, which were taught by the author for more than 20 years at the Faculty of Radiophysics at Nizhny Novgorod State University (NNSU).

The aim of the course was not only to express fundamental ideas and methods of the theory of oscillations as a science of evolutionary processes, but also to teach the audience the methods and techniques of solving specific (practical) problems.

The key role in forming this lecture course is played by qualitative methods of the theory of dynamical systems and methods of the theory of bifurcations, which follow the tradition of Nizhny Novgorod school of nonlinear oscillations. These methods are even used when solving simple problems, where, in principle, their use is not necessary. Such a way of presenting the following material allows us, first of all, to reveal the essence and fundamental principles of the methods, and, secondly, for the reader to develop the skills necessary to put them to use, which appears to be important for the transition to studying more complex problems.

The book is constructed in the form of lectures in accordance with the syllabus of the course "Theory of Oscillations" for the Faculty of Radiophysics at NNSU. Yet, the content of nearly every lecture in this book is expanded further than it is usually presented during the reading of a formal lecture. This makes it possible for the reader to gain additional knowledge on the subject. At the end of each lecture, there are test questions and problems for revision and independent study.

This text could also prove useful to undergraduate and graduate students specializing in the field of nonlinear dynamics, information systems, control theory, biophysics, and so on.

The author is grateful to the colleagues at the department of "Theory of Oscillations and Automated Control" for many useful discussions on the topics of this text and to the colleagues from the department of Nonlinear Dynamics at the Institute of Applied Physics of the Russian Academy of Sciences.

Nizhny Novgorod *Vladimir I. Nekorkin*
October 2014

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Introduction to the Theory of Oscillations

1.1
General Features of the Theory of Oscillations

Oscillatory processes and systems are so widely distributed in nature, technology, and society that we frequently encounter them in our everyday life and can, apparently, formulate their basic properties without difficulty. Indeed, when we hear about fluctuations in temperature, exchange rates, voltage, a pendulum, the water level, and so on, we understand that it is in relation to processes in time or space, which have varying degrees of repetition and return to their original or similar states. Moreover, these base properties of the processes do not depend on the nature of systems and Can, therefore, be described and studied from just the point of view of a general interdisciplinary approach. This is exactly the approach that the theory of oscillations explores, the subject of which are the oscillatory phenomena and the processes in systems of different nature. The theory of oscillations gets its oscillatory properties from the analysis of the corresponding models. As a result of such an analysis, a connection between the parameters of the model and its oscillatory properties is established.

The theory of oscillations is both an applied and fundamental science. The applied character of the theory of oscillations is determined by its multiple applications in physics, mechanics, automated control, radio engineering and electronics, instrumentation, and so on. In these spheres of science, a large amount of research of different systems and phenomena was carried out, using the methods of the theory of oscillations. Furthermore, new technical directions have arisen on the basis of the theory of oscillations, namely, vibrational engineering and vibrational diagnostics, biomechanics, and so on. The fundamental characteristic of the theory of oscillations is based on the studied models themselves. They are the so-called dynamical systems, with the help of which one can describe any determinate evolution in time or in time and space. It is exactly the study of dynamical systems that allowed the theory of oscillations to introduce the concepts and conditions, develop the methods, and achieve the results that exert a large influence on other natural sciences. Here, we only mention the linearized stability theory, the concept of self-sustained oscillations and resonance, bifurcation theory, chaotic oscillations, and so on.

Introduction to Nonlinear Oscillations, First Edition. Vladimir I. Nekorkin.

1.2
Dynamical Systems

Consider the system, the state of which is determined by the vector $\mathbf{x}(t) \in \mathrm{R}^n$. Assume that the evolution of the system is determined by a single parameter family of operators $G^t, t \in \mathrm{R}$ (or $t \in \mathrm{R}_+$) or $t \in \mathrm{Z}$ (or $t \in \mathrm{Z}_+$), such that the state of the system at the instant t

$$\mathbf{x}(t, \mathbf{x}_0) = G^t \mathbf{x}_0 \tag{1.1}$$

where $\mathbf{x}_0$ is its initial state (initial point). We also assume that the evolutionary operators satisfy the following two properties, which reflect the deterministic character of the described processes.

The first property: G^0 is the identity operator, that is,

$$\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0, \tag{1.2}$$

for any $\mathbf{x}_0$. This property means that the state of the system cannot change spontaneously.

The second property of the evolutionary operators is

$$G^{t_1+t_2} = G^{t_1} \cdot G^{t_2} = G^{t_2} \cdot G^{t_1}, \tag{1.3}$$

that is,

$$\mathbf{x}(t_1 + t_2, \mathbf{x}_0) = \mathbf{x}(t_1, \mathbf{x}(t_2, \mathbf{x}_0)) = \mathbf{x}(t_2, \mathbf{x}(t_1, \mathbf{x}_0)) \tag{1.4}$$

According to (1.3), the system reaches the same final state, regardless of whether it does so within one time interval $t_1 + t_2$ or over several successive intervals t_1 and t_2, equal in sum to $t_1 + t_2$.

The combination of all initial points • or of all possible states of the system (in this case, $X = \mathrm{R}^n$) is called a *phase space*, and a pair $(X, \{G^t\})$, where a family of evolutionary operators satisfies the conditions (1.2) and (1.3), is a dynamical system.

Dynamical systems are divided into two important categories, one with continuous time if $t \in \mathrm{R}$ or R_+ and another with discrete time if $t \in \mathrm{Z}$ or Z_+.

The evolution of the system corresponds to the motion of the representation point in the phase space along the trajectory $\Gamma = \bigcup_t G^t \mathbf{x}_0$. The family $\Gamma^+ = \bigcup_{t \geq 0} G^t \mathbf{x}_0 \left(\Gamma^- = \bigcup_{t<0} G^t \mathbf{x}_0 \right)$ is called a positive semi-trajectory going through the initial point $\mathbf{x}_0$. If the family$\{G^t\}$ is continuous at t (for dynamical systems with continuous time), then the trajectories (semi-trajectory) represent continuous curves at X. For the dynamical systems with discrete time, the trajectories are discrete subsets in the phase space.

Let us introduce the idea of the invariance of a set, which will be necessary in what follows. The set $A \subset X$ is called positively (negatively) invariant if it consists of positive (negative) semi-trajectories, that is, A is positively (negatively) invariant if $G^t A \subset A, t > 0 (t < 0)$. The set A is called invariant if it is invariant both when positive and when negative.

1.2.1
Types of Trajectories

Let us define the main types of the dynamical system trajectories.

1) The point $\mathbf{x}_0$ is called a fixed point of a dynamical system if $G^t\mathbf{x}_0 = \mathbf{x}_0$ for all t (for systems with continuous time, such points are more often called *equilibrium points*).
2) The point $\mathbf{x}_0$ is called periodic if there exists $T > 0$, such that $G^T\mathbf{x}_0 = \mathbf{x}_0$ and $G^t\mathbf{x}_0 \neq \mathbf{x}_0$ for $0 < t < T$, and its corresponding trajectory $\bigcup\limits_{0\leq t\leq T} G^t\mathbf{x}_0$ of the dynamical system passing through this point is periodic. A periodic trajectory is a closed curve in the phase space of a dynamical system with continuous time or a set of T-periodic points for the dynamical systems with discrete time.
3) The point $\mathbf{x}_0$ is called nonwandering if for any open set $U \ni \mathbf{x}_0$ of this point and any $t_0 > 0$ there exists $t > t_0$, such that $G^tU \bigcap U \neq \emptyset$. The trajectory going through a nonwandering point is called a nonwandering trajectory.

There is a correspondence between the trajectories of dynamical systems and the motions of real systems. Stationary states of real systems correspond to fixed points of dynamical systems, periodic motions correspond to periodic trajectories, and the motions with some degree of repetition of their states in time correspond to nonwandering trajectories.

Note that the aforementioned trajectories can also exist in the dynamical systems whose phase space is not necessarily R^n. For example, the phase space of a dynamical system describing the oscillations of a mathematical pendulum is a cylinder, $X = \mathrm{S}^1 \times \mathrm{R}$, as the state of the pendulum at any moment of time is uniquely described by its phase $\varphi(t)$ determined with accuracy up to $2\pi(\varphi \in \mathrm{S}^1)$ and by the value of its velocity $\dot{\varphi} \in \mathrm{R}$.

1.2.2
Dynamical Systems with Continuous Time

For many dynamical systems with continuous time, the rule, which allows one to find the state at any point in time according to the initial state, is shown by the following system of ordinary differential equations:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_N), \quad i = 1, 2, \ldots, N$$

or, in vector form,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \mathbf{x} \in \mathrm{R}^\mathrm{n}, \mathbf{F} : \mathrm{R}^n \to \mathrm{R}^n, \tag{1.5}$$

for which the conditions of existence and uniqueness of the solutions are satisfied (hereafter we denote differentiation in time by an overdot). In this case, the family $G^t\mathbf{x}_0$ is simply given by the solution of system (1.5) with the initial condition $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$. For example, for the linear system

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where A is an $n \times n$ matrix with constant elements, the solution has the form $\mathbf{x}(t, \mathbf{x}_0) = e^{At}\mathbf{x}_0$, where e^{At} is an $n \times n$ matrix. As the matrices e^{At_1} and e^{At_2} commute for any pair t_1, t_2, the property (1.3)

$$e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = e^{At_2} \cdot e^{At_1}$$

is fulfilled. Evidently, the property (1.2) is also fulfilled.

In another example, we consider the system given in polar coordinates

$$\dot{\rho} = \lambda\rho, \quad \dot{\varphi} = \omega,$$

where ρ and ω are the parameters. The solution of this system has the following form:

$$\rho = \rho_0 e^{\lambda t}, \quad \varphi = \omega t + \varphi_0$$

Hence, the evolution operators are specified as follows:

$$G^t : (\rho_0, \varphi_0) \to (\rho_0 e^{\lambda t}, \omega t + \varphi).$$

Evidently, the properties (1.2) and (1.3) are fulfilled.

Note that the right-hand side of system (1.5) does not depend on time explicitly. Such systems are called *autonomous*. There is also a large number of problems (e.g., systems subjected to an alternating external force), which are described by dynamical systems whose right-hand sides depend on time explicitly. They are called *nonautonomous*.

1.2.3
Dynamical Systems with Discrete Time

Dynamical systems with discrete time are usually defined as follows:

$$\mathbf{x}(n+1) = \mathbf{F}(\mathbf{x}(n)), \tag{1.6}$$

where $\mathbf{F} : \mathrm{R}^n \to \mathrm{R}^n$ is the map and $n \in \mathrm{Z}_+ = \{0, 1, 2, ...\}$ is the discrete time.

For such systems, a trajectory is a finite or countable set of points in R^n. Another equivalent notation is also used sometimes for a dynamical system with discrete time:

$$\overline{\mathbf{x}} = \mathbf{F}(\mathbf{x}),$$

where $\overline{\mathbf{x}}$ is the image of the point å under the action of the map $\mathbf{F}$. In this manual, we will use both forms of notation of maps.

Let us illustrate the concept of a dynamical system with discrete time by using the example of a one-dimensional map,

$$\overline{x} = 2x, \ \bmod 1 \tag{1.7}$$

The phase space of this map is the interval $[0, 1]$. Let $x(0) = 1/5$. Directly from (1.7), we obtain

$$x(0) = \frac{1}{5} \to x(1) = \frac{2}{5} \to x(2) = \frac{4}{5} \to x(3) = \frac{3}{5} \to x(4) = \frac{1}{5}$$