METHODS OF APPLIED MATHEMATICS

By

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Preface

The principal aim of this volume is to place at the disposal of the engineer or physicist the basis of an intelligent working knowledge of a number of facts and techniques relevant to four fields of mathematics which usually are not treated in courses of the "Advanced Calculus" type, but which are useful in varied fields of application. The text includes the result of a series of revisions of material originally prepared in mimeographed form for use at the Massachusetts Institute of Technology.

Account is taken of the fact that most students in the fields of application have neither the time nor the inclination for the study of elaborate treatments of each of these topics from the classical point of view. At the same time it is realized that efficient use of facts or techniques depends strongly upon a substantial understanding of the basic underlying principles. For this reason, care has been taken throughout the text either to provide rigorous proofs, when it is believed that those proofs can be readily comprehended by a wide class of readers, or to state the desired results as precisely as possible and indicate why those results might have been *formally* anticipated.

In each chapter, the treatment consists in showing how typical problems may arise, in establishing those parts of the relevant theory which are of principal practical significance, and in developing techniques for analytical and numerical analysis and problem solving.

Whereas experience gained from a course on the Advanced Calculus level is presumed, the treatments are almost completely self-contained, so that the nature of this preliminary course is not of great importance.

In order to increase the usefulness of the volume as a basic or supplementary text, and as a reference volume, an attempt has been made to organize the material so that there is very little essential interdependence among the chapters, and so that considerable flexibility exists with regard to the omission of topics within chapters. In addition, a large amount of supplementary material is included in annotated problems which complement numerous exercises, of varying difficulty, which are arranged in correspondence with successive sections of the text at the ends of the chapters. Answers to all problems either are incorporated into their statement or are listed at the end of the book.

The first chapter deals principally with *linear algebraic equations*, *quadratic* and *Hermitian forms*, and operations with *vectors* and *matrices*, with special emphasis on the concept of characteristic values. A brief summary of corresponding results in *function space* is included for comparison, and for convenient reference. Whereas a considerable amount of material is presented, particular care was taken here to order and even to overlap the demonstrations in such a way that maximum flexibility in selection of topics is present.

The first portion of the second chapter deals carefully with the variational notation and derives the Euler equations relevant to a large class of problems in the *calculus of variations*. More than usual emphasis is placed on the significance of natural boundary conditions. Generalized coordinates, Hamilton's principle, and Lagrange's equations are treated and illustrated within the framework of this theory. The chapter concludes with a discussion of the formulation of minimal principles of more general type, and with the application of direct and semidirect methods of the calculus of variations to the exact and approximate solution of practical problems.

The third chapter combines the presentation of available methods for solving the simpler types of *difference equations* with a description of the application of *finite-difference methods* to the approximate solution of problems governed by partial differential equations, and includes consideration of the troublesome problems of convergence and stability. Much of this material, the importance of which has increased greatly with modern developments in the field of numerical analysis, has not appeared previously in integrated form.

The concluding chapter deals with the formulation and theory of linear *integral equations*, and with exact and approximate methods for obtaining their solutions, particular emphasis being placed on the several equivalent interpretations of the relevant Green's function. Considerable supplementary material is provided in annotated problems.

Many compromises between mathematical elegance and practical significance were found to be necessary. It is hoped that the present volume will serve to ease the way of the engineer or physicist into the more advanced areas of applicable mathematics, for which his need is steadily increasing, without obscuring from him the existence of certain *difficulties* often implied by the phrase "It can be shown," and without failing to warn him of certain *dangers* involved in formal application of techniques beyond the limits inside which their validity has been well established.

The author is indebted to colleagues and students in various fields for help in selecting and revising the content and presentation, and particularly to Professor A. A. Bennett for many valuable criticisms and suggestions.

F. B. HILDEBRAND

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CHAPTER ONE

Matrices, Determinants, and Linear Equations

1.1. Introduction. In many fields of analysis we find it necessary to deal with an *ordered set* of elements, which may be numbers or functions. In particular, we may deal with an ordinary *sequence* of the form $a_1, a_2, \ldots, a_n$, or with a two-dimensional *array* such as the rectangular arrangement

$$\begin{matrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22}, & \cdots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}, & a_{m2}, & \cdots, & a_{mn}, \end{matrix}$$

consisting of m rows and n columns.

When suitable laws of equality, addition and subtraction, and multiplication are associated with sets of such rectangular arrays, the arrays are called *matrices*, and are then designated by a special symbolism. The laws of combination are specified in such a way that the matrices so defined are of frequent usefulness in both practical and theoretical considerations.

Since matrices are perhaps most intimately associated with sets of *linear algebraic equations*, it is desirable to investigate the general nature of the solutions of such sets of equations by elementary methods, and hence to provide a basis for certain definitions and investigations which follow.

1.2. Linear equations. The Gauss-Jordan reduction. We deal first with the problem of attempting to obtain solutions of a set of m linear equations in n unknown variables $x_1, x_2, \ldots, x_n$, of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = c_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m \end{array}\right\}, \tag{1}$$

by direct calculation.

Under the assumption that (1) does indeed possess a solution, the *Gauss-Jordan reduction* proceeds as follows:

First Step. Suppose that $a_{11} \neq 0$. (Otherwise, renumber the equations or variables so that this is so.) Divide both sides of the first equation by a_{11}, so that the resultant equivalent equation is of the form

$$x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n = c'_1. \tag{2}$$

Multiply both sides of (2) successively by $a_{21}, a_{31}, \ldots, a_{m1}$, and subtract the respective resultant equations from the second, third, . . . , mth equations of (1), to reduce (1) to the form

$$\left.\begin{array}{r} x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n = c'_1, \\ a'_{22}x_2 + \cdots + a'_{2n}x_n = c'_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a'_{m2}x_2 + \cdots + a'_{mn}x_n = c'_m \end{array}\right\}. \tag{3}$$

Second Step. Suppose that $a'_{22} \neq 0$. (Otherwise, renumber the equations or variables so that this is so.) Divide both sides of the second equation in (3) by a'_{22}, so that this equation takes the form

$$x_2 + a''_{23}x_3 + \cdots + a''_{2n}x_n = c''_2, \tag{4}$$

and use this equation, as in the first step, to eliminate the coefficient of x_2 in *all other equations* in (3), so that the set of equations becomes

$$\left.\begin{array}{r} x_1 \quad + a''_{13}x_3 + \cdots + a''_{1n}x_n = c''_1, \\ x_2 + a''_{23}x_3 + \cdots + a''_{2n}x_n = c''_2, \\ a''_{33}x_3 + \cdots + a''_{3n}x_n = c''_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a''_{m3}x_3 + \cdots + a''_{mn}x_n = c''_m \end{array}\right\}. \tag{5}$$

Remaining Steps. Continue the above process r times until it terminates, that is, until $r = m$ *or* until all coefficients of the x's are *zero* in all equations following the rth equation. We shall speak of these $m - r$ equations as the *residual equations.*

There then exist two alternatives. *First,* it may happen that one or more of the residual equations has a nonzero right-hand member, and hence is of the form $0 = c_k^{(r)}$ (where in fact $c_k^{(r)} \neq 0$). In this case, the assumption that a solution of (1) exists leads to a *contradiction,* and hence *no solution exists.* The set (1) is then said to be *inconsistent* or *incompatible.*

Otherwise, no contradiction exists, and the set (1) of m equations is reduced to an equivalent set of r equations which, after a transposition, can be written in the form

$$\left.\begin{aligned} x_1 &= \gamma_1 + \alpha_{11}x_{r+1} + \cdots + \alpha_{1,n-r}x_n, \\ x_2 &= \gamma_2 + \alpha_{21}x_{r+1} + \cdots + \alpha_{2,n-r}x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_r &= \gamma_r + \alpha_{r1}x_{r+1} + \cdots + \alpha_{r,n-r}x_n \end{aligned}\right\}, \tag{6}$$

where the γ's and α's are definite constants related to the coefficients in (1). Hence, in this case the most general solution of (1) expresses each of the r variables $x_1, x_2, \ldots, x_r$ as a definite constant plus a definite linear combination of the remaining $n - r$ variables, each of which can be assigned *arbitrarily.*

If $r = n$, a *unique* solution is obtained. Otherwise, we say that an *$(n - r)$-fold infinity* of solutions exists. The number $n - r = d$ may be called the *defect* of the system (1). We notice that if the system (1) is consistent and r is less than m, then $m - r$ of the equations (namely, those which correspond to the residual equations) are actually ignorable, since they are implied by the remaining r equations.

The reduction may be illustrated by considering the four simultaneous equations

$$\left.\begin{aligned} x_1 + 2x_2 - x_3 - 2x_4 &= -1, \\ 2x_1 + x_2 + x_3 - x_4 &= 4, \\ x_1 - x_2 + 2x_3 + x_4 &= 5, \\ x_1 + 3x_2 - 2x_3 - 3x_4 &= -3 \end{aligned}\right\}. \tag{7}$$

It is easily verified that after two steps in the reduction one obtains the equivalent set

$$\left.\begin{aligned} x_1 \quad + x_3 \qquad &= 3, \\ x_2 - x_3 - x_4 &= -2, \\ 0 &= 0, \\ 0 &= 0 \end{aligned}\right\}.$$

Hence the system is of defect *two*. If we write $x_3 = c_1$ and $x_4 = c_2$, it follows that the general solution can be expressed in the form

$$x_1 = 3 - c_1, \quad x_2 = -2 + c_1 + c_2, \quad x_3 = c_1, \quad x_4 = c_2, \tag{8a}$$

where c_1 and c_2 are arbitrary constants. This two-parameter family of solutions can also be written in the symbolic form

$$\{x_1, x_2, x_3, x_4\} = \{3, -2, 0, 0\} + c_1\{-1, 1, 1, 0\} + c_2\{0, 1, 0, 1\}. \tag{8b}$$

It follows also that the third and fourth equations of (7) must be consequences of the first two equations. Indeed, the third equation is obtained by subtracting the first from the second, and the fourth by subtracting one-third of the second from five-thirds of the first.

The Gauss-Jordan reduction is useful in actually obtaining numerical solutions of sets of linear equations,* and it has been presented here also for the purpose of motivating certain definitions and terminologies which follow.

1.3. Matrices. The set of equations (1) can be visualized as a *linear transformation* in which the set of n numbers $\{x_1, x_2, \ldots,$

* In place of eliminating x_k from *all* equations except the kth, in the kth step, one may eliminate x_k only in those equations *following* the kth equation. When the process terminates, after r steps, the rth unknown is given explicitly by the rth equation. The $(r - 1)$th unknown is then determined by substitution in the $(r - 1)$th equation, and the solution is completed by working back in this way to the first equation. The method just outlined is associated with the name of *Gauss*. In order that the "round-off" errors be as small as possible, it is desirable that the sequence of eliminations be ordered such that the coefficient of x_k in the equation used to eliminate x_k is as large as possible in absolute value, relative to the remaining coefficients in that equation.

A modification of this method, due to Crout (Reference 7), which is particularly well adapted to the use of desk computing machines, is described in an appendix.

$x_n\}$ is transformed into the set of m numbers $\{c_1, c_2, \ldots, c_m\}$. The transformation is clearly specified by the coefficients a_{ij}.

The rectangular array of these coefficients, usually enclosed in square brackets,

$$\mathbf{a} \equiv [a_{ij}] \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{9}$$

which consists of m *rows* and n *columns* of *elements*, is called an $m \times n$-*matrix* when certain laws of combination, yet to be specified, are laid down. In the symbol a_{ij}, representing a typical element, the *first* subscript (here i) denotes the row and the *second* subscript (here j) the column occupied by the element.

The sets of quantities x_i $(i = 1, 2, \ldots, n)$ and c_i $(i = 1, 2, \ldots, m)$ are conventionally represented as matrices of *one column* each. In order to emphasize the fact that a matrix consists of only one column, it is convenient to indicate it by braces, rather than brackets, and so to write

$$\mathbf{x} \equiv \{x_i\} \equiv \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix}, \qquad \mathbf{c} \equiv \{c_i\} \equiv \begin{Bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{Bmatrix}. \tag{10a,b}$$

For convenience in writing, the elements of a one-column matrix are frequently arranged horizontally, the use of braces then serving to indicate the transposition.

If we visualize (1) as stating that the matrix $\mathbf{a}$ transforms the one-column matrix $\mathbf{x}$ into the one-column matrix $\mathbf{c}$, it is natural to write the transformation in the form

$$\mathbf{a}\,\mathbf{x} = \mathbf{c}, \tag{11}$$

where $\mathbf{a} = [a_{ij}]$, $\mathbf{x} = \{x_i\}$, and $\mathbf{c} = \{c_i\}$.

On the other hand, the set of equations (1) can be written in the form

$$\sum_{k=1}^{n} a_{ik}x_k = c_i \qquad (i = 1, 2, \cdots, m), \tag{12}$$

which leads to the matrix equation

$$\left\{\sum_{k=1}^{n} a_{ik}x_k\right\} = \{c_i\}. \tag{12a}$$

Hence, if (11) and (12a) are to be equivalent, we are led to the *definition*

$$\mathbf{a}\,\mathbf{x} = [a_{ik}]\{x_k\} \equiv \left\{\sum_{k=1}^{n} a_{ik}x_k\right\}. \tag{13}$$

Formally, we merely replace the *column* subscript in the general term of the *first* factor by a new *dummy index k*, and replace the *row* subscript in the general term of the *second* factor by the same dummy index, and sum over that index.

The definition is clearly applicable only when the number of *columns* in the *first* factor is equal to the number of *rows* (elements) in the *second* factor. Unless this condition is satisfied, the product is undefined.

We notice that a_{ik} is the element in the ith row and kth column of $\mathbf{a}$, and that x_k is the kth element in the one-column matrix $\mathbf{x}$. Since i ranges from 1 to m in a_{ij}, the definition (13) states that the product of an $m \times n$-matrix into an $n \times 1$-matrix is an $m \times 1$-matrix (m elements in one column). The ith element in the product is obtained from the ith row of the first factor and the single column of the second factor, by multiplying together the first elements, second elements, and so forth, and adding these products together algebraically.

Thus, for example, the definition leads to the result

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 1\cdot 1 + 0\cdot 2 \\ 2\cdot 1 + 1\cdot 2 \\ -1\cdot 1 + 2\cdot 2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 3 \end{Bmatrix}.$$

Now suppose that the n variables $x_1, \ldots, x_n$ are expressed as linear combinations of s new variables $y_1, \ldots, y_s$, that is, that a set of relations holds of the form

$$x_i = \sum_{k=1}^{s} b_{ik}y_k \qquad (i = 1, 2, \cdots, n). \tag{14}$$

If the original variables satisfy (12), the equations satisfied by the new variables are obtained by introducing (14) into (12). In addition to replacing i by k in (14), for this introduction, we must clearly replace k in (14) by a *new* dummy index, say l, to avoid ambiguity of notation. The result of the substitution then takes the form

$$\sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{s} b_{kl} y_l \right) = c_i \qquad (i = 1, 2, \cdots, m), \tag{15a}$$

or, since the order in which the finite sums are formed is immaterial,

$$\sum_{l=1}^{s} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) y_l = c_i \qquad (i = 1, 2, \cdots, m). \tag{15b}$$

In matrix notation, the transformation (14) takes the form

$$\mathbf{x} = \mathbf{b}\,\mathbf{y} \tag{16}$$

and, corresponding to (15a), the introduction of (16) into (11) gives

$$\mathbf{a}(\mathbf{b}\,\mathbf{y}) = \mathbf{c}. \tag{17}$$

But if we write

$$p_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad \begin{pmatrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, s \end{pmatrix} \tag{18}$$

equation (15b) takes the form

$$\sum_{l=1}^{s} p_{il} y_l = c_i \qquad (i = 1, 2, \cdots, m),$$

and hence, in accordance with (12) and (13), the matrix form of the transformation (15b) is

$$\mathbf{p}\,\mathbf{y} = \mathbf{c}. \tag{19}$$

Thus it follows that the result of operating on **y** by **b**, and on the product by **a** [given by the left-hand member of (17)], is the same as the result of operating on **y** directly by the matrix **p**. We accordingly *define* this matrix to be the product **a b**,

$$\mathbf{a}\,\mathbf{b} = [a_{ik}][b_{kj}] \equiv \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]. \tag{20}$$

Recalling that the first subscript in each case is the row index and the second the column index, we see that if the first factor has

m rows and n columns, and the second n rows and s columns, the index i in the *right-hand* member may vary from 1 to m while the index j in that member may vary from 1 to s. Hence, the *product of an $m \times n$-matrix into an $n \times s$-matrix is an $m \times s$-matrix.* The element p_{ij} in the ith row and jth column of the product is formed by multiplying together corresponding elements of the ith *row* of the *first* factor and the jth *column* of the *second* factor, and adding the results algebraically.

Thus, for example, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1\cdot 1 + 0\cdot 1 + 1\cdot 2)(1\cdot 2 + 0\cdot 0 + 1\cdot 1)(1\cdot 1 + 0\cdot 1 + 1\cdot 0) \\ (1\cdot 1 - 2\cdot 1 + 1\cdot 2)(1\cdot 2 - 2\cdot 0 + 1\cdot 1)(1\cdot 1 - 2\cdot 1 + 1\cdot 0) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

We notice that $\mathbf{a}\,\mathbf{b}$ is defined only if the number of *columns* in $\mathbf{a}$ is equal to the number of *rows* in $\mathbf{b}$. In this case, the two matrices are said to be *conformable* in the order stated.

If $\mathbf{a}$ is an $m \times n$-matrix and $\mathbf{b}$ an $n \times m$-matrix, then $\mathbf{a}$ and $\mathbf{b}$ are conformable in either order, the product $\mathbf{a}\,\mathbf{b}$ then being a *square* matrix of order m and the product $\mathbf{b}\,\mathbf{a}$ a square matrix of order n. Even in the case when $\mathbf{a}$ and $\mathbf{b}$ are square matrices of the same order the products $\mathbf{a}\,\mathbf{b}$ and $\mathbf{b}\,\mathbf{a}$ are not generally equal. For example, in the case of two square matrices of order two we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix},$$

and also

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} & a_{12}b_{11} + a_{22}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} & a_{12}b_{21} + a_{22}b_{22} \end{bmatrix}.$$

Thus, in multiplying $\mathbf{b}$ by $\mathbf{a}$ in such cases, we must carefully distinguish *pre*multiplication ($\mathbf{a}\,\mathbf{b}$) from *post*multiplication ($\mathbf{b}\,\mathbf{a}$).

The *sum* of two $m \times n$-matrices $[a_{ij}]$ and $[b_{ij}]$ is defined to be the matrix $[a_{ij} + b_{ij}]$. Further, the product of a number k and a