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Second Course in Ordinary Differential Equations for Scientists and Engineers

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PREFACE

The world abounds with introductory texts on ordinary differential equations and rightly so in view of the large number of students taking a course in this subject. However, for some time now there is a growing need for a junior-senior level book on the more advanced topics of differential equations. In fact the number of engineering and science students requiring a second course in these topics has been increasing. This book is an outgrowth of such courses taught by us in the last ten years at Worcester Polytechnic Institute.

The book attempts to blend mathematical theory with nontrivial applications from various disciplines. It does not contain lengthy proofs of mathematical theorems as this would be inappropriate for its intended audience. Nevertheless, in each case we motivated these theorems and their practical use through examples and in some cases an "intuitive proof" is included. In view of this approach the book could be used also by aspiring mathematicians who wish to obtain an overview of the more advanced aspects of differential equations and an insight into some of its applications. We have included a wide range of topics in order to afford the instructor the flexibility in designing such a course according to the needs of the students. Therefore, this book contains more than enough material for a one semester course.

The text begins with a review chapter (Chapter 0) which may be omitted if the students have just recently completed an introductory course. Chapter 1 sets the stage for the study of Boundary Value Problems, which are not normally included in elementary texts. Examples of important boundary value problems are covered in Chapter 2. Although systems of ordinary differential equations are contained in beginning texts, a more detailed approach to such problems is discussed in Chapter 3. Chapters 4 through 10 examine specific applications of differential equations such as Perturbation Theory, Stability, Bifurcations. (See Table of Contents.)

A word about the numbering system used in this book. The sections of each chapter are numbered 1,2,3... Occasionally there are sub-sections numbered 1.1, 1.2,... Definitions, Theorems, Lemmas, Corollaries, Examples start at 1 in each section. Formulas are tied to sections e.g. (2.1), (2.2) are in section 2. Exercises start at 1 in each chapter.

Finally, special thanks are due to Mrs. C. M. Lewis who typed the manuscript. Her efforts went well beyond the call of duty and she spared no time or effort to produce an accurate and esthetically appealing "camera ready" copy of the book. Thanks are also due to the staff at Springer Verlag for their encouragement and guidance.

CONTENTS

CHAPTER 0. REVIEW

1. SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS BY SERIES

1.1 INTRODUCTION.

Probably one of the least understood topics in a beginning ordinary differential equations course is finding a series solution of a given equation. Unfortunately this lack of knowledge hinders a student's understanding of such important functions as Bessel's function, Legendre polynomials and other such functions which arise in modern engineering problems.

In this chapter we shall undertake a general review of the ideas behind solving ordinary differential equations by the use of infinite series. We shall first examine Taylor series solutions which can be expanded about ordinary points. If the point about which we wish to find the expansion is singular (i.e. not ordinary), we go on to investigate the types of singular points and finally discuss how, under certain conditions, a Frobenius series applies to them. Although our discussion will be limited to second order differenital equations, these methods can be applied to higher order equations.

1.2 ORDINARY AND SINGULAR POINTS.

We recall that a second order linear ordinary differential equation can be written in the form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x)$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$, $f(x)$ are assumed to be continuous real-valued functions on an interval I.

If $f(x) = 0$, the equation is said to be homogeneous and it is to the solution of such types of equations we shall devote our attention, that is, in what follows we shall look at methods used to solve the equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad .$$

The first step in solving an equation by use of series is to determine whether the point $x = x_0$ about which we wish to expand the series is ordinary or singular. This is done by dividing both sides of the equation by $a_0(x)$ so that our equation looks like

$$y'' + \frac{a_1(x)}{a_0(x)} y' + \frac{a_2(x)}{a_0(x)} y = 0$$

If $\frac{a_1(x)}{a_0(x)}$ and $\frac{a_2(x)}{a_0(x)}$ can be expanded in a Taylor series about $x = x_0$, then $x = x_0$ called an *ordinary point*. If this is not the case then $x = x_0$ is called a *singular* point.

In many situations $\frac{a_1(x)}{a_0(x)}$ and $\frac{a_2(x)}{a_0(x)}$ are rational functions and it is known that such a function has a Taylor series expansion about all points $x \in I$ except where the denominator vanishes.

Example 1: If every coefficient of the differential equation is a constant, then every value of x is an ordinary point.

Example 2: What are the ordinary points of

$$x^2(x^2-1)y'' + (x+1)y' + xy = 0.$$

First we rewrite the equation in normal form

$$y'' + \frac{1}{x^2(x-1)}\, y' + \frac{1}{x(x^2-1)}\, y = 0.$$

Since the denominators of coefficients vanish at $x = 0, +1, -1$ these are singular points. All other values of x are ordinary points.

Example 3: Examine

$$xy'' + \sin xy' + x^2y = 0$$

for singular and ordinary points.

Rewriting the equation we have

$$y'' + \frac{\sin x}{x}\, y' + xy = 0.$$

The Taylor series expansion of the y'-coefficient is

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}.$$

We can use the ratio test to easily show $\frac{\sin x}{x}$ converges for all x. Therefore, every point is an ordinary point.

More will be said concerning singular points in Section 2.

EXERCISE 1

1. List all ordinary and singular points
   (a) $y'' + 2y' + y = 0$
   (b) $y'' - xy = 0$
   (c) $xy'' - y = 0$
   (d) $x(x-1)y'' + (x+1)y' + (x+2)y = 0$
   (e) $(x^2+x-6)y'' + (x^2+x+1)y' + (x-1)y = 0$
2. List all ordinary and singular points
   (a) $y'' + \sin xy' + e^xy = 0$
   (b) $3\sqrt{x+1}\, y'' - xy' + 3\sqrt{x-1}\, y = 0$

(c) $\sin xy'' + \cos xy' + xy = 0,\ x \in [-2\pi, 2\pi]$

(d) $y'' + \sum_{n=0}^{\infty} \frac{x^n}{n!} y' + \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} y = 0$

1.3 TAYLOR SERIES SOLUTIONS.

Once we have ascertained that $x = x_0$ is an ordinary point then we know we can find a solution in a Taylor series expanded about $x = x_0$, that is there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x-x_0)^n \,. \tag{1.1}$$

It is our job to find the values of the coefficients c_n which give us a solution. This is done by substituting the series (1.1) into the given differential equation. The method can be best explained by example.

Example 4: Find the Taylor series solution of

$$y'' + xy' + 2y = 0 \tag{1.2}$$

about $x = 0$.

Step 1: Obviously $x = 0$ is an ordinary point so we assume a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \,. \tag{1.3}$$

Differentiating twice with respect to x we find

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \tag{1.4}$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} \,. \tag{1.5}$$

Notice that the first term in the series y' and the first two terms in the series y'' are zero, therefore we can rewrite (1.4) and (1.5) in the form

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \tag{1.6}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} . \tag{1.7}$$

Substituting (1.3), (1.6), and (1.7) into equation (1.2) we have

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} nc_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \tag{1.8}$$

or

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0.$$

Step 2: Reindex individual series so that all powers of x are the same. Since two series are already of the form x^n we shall change the first series on the lefthand side. Let n be replaced by $n + 2$, i.e.

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n+2=2}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n .$$

Using this result in (1.8) we obtain the equation

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0. \tag{1.9}$$

Step 3: We notice now that although each series contains the term x^n, they do not start at the same time. The series start at $n = 0, 1, 0$ repectively. Our goal is to combine all three series into one series but since some series contain terms not present in others we must make the following adjustment. Bring out from under the Σ sign those terms in a series which do not appear in all of them. That is, we write (1.9) in the form

$$2c_2 + \sum_{n=1}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$+ \sum_{n=1}^{\infty} nc_n x^n + 2c_0 + \sum_{n=1}^{\infty} 2c_n x^n = 0. \tag{1.10}$$

Now since each series starts at the same point (i.e. $n = 1$) and has the same power (i.e. x^n), we can combine them into one series. Therefore, we can write (1.10) as

$$2c_2 + 2c_0 + \sum_{n=1}^{\infty} \{(n+2)(n+1)c_{n+2} + (n+2)c_n\}x^n = 0.$$

Step 4: Now if a series converges over an interval to the value zero, it is necessary that every coefficient of x^n be zero. Applying this requirement to our problem we have

$$2c_2 + 2c_0 = 0 \quad \text{or} \quad c_2 = -c_0 \qquad (1.11)$$

and

$$(n+2)(n+1)c_{n+2} + (n+2)c_n = 0, \; n \geq 1 .$$

It is convenient to solve the second equation above for the higher coefficient in terms of the lower one. Thus

$$c_{n+2} = -\frac{c_n}{n+1}, \; n \geq 1 .$$

This equation is known as the *recurrence relation* and it is by using this formula repeatedly that we get the desired coefficients. (Notice that the *recurrence relation* yields equation (1.11) for $n = 0$ but this is not always true!).

Step 5: To find the general solution we must find two linearly independent solutions from expression (1.3). Usually this is done by stating that the lowest arbitrary coefficient is not zero while the next larger arbitrary coefficient is zero. A second linearly independent solution can be found by interchanging the above conditions. With this in mind let $c_0 \neq 0$, $c_1 = 0$, then

$$c_2 = -c_0$$

$$c_3 = -\frac{c_1}{2} = 0$$

$$c_4 = -\frac{c_2}{3} = \frac{c_0}{3}$$

$$c_5 = 0$$

$$c_6 = -\frac{c_4}{5} = -\frac{c_0}{3\cdot5}$$

which gives us one solution

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

$$= c_0 - c_0 x^2 + \frac{c_0}{3} x^4 - \frac{c_0}{3\cdot5} x^6$$

$$= c_0\left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{3\cdot5} + \ldots\right) .$$

To find the second solution let $c_0 = 0$, $c_1 \neq 0$. Then

$$c_0 = 0$$

$$c_1 \neq 0$$

$$c_2 = -c_0 = 0$$

$$c_3 = -\frac{c_1}{2}$$

$$c_4 = 0$$

$$c_5 = -\frac{c_3}{4} = \frac{c_1}{2\cdot4}$$

$$c_6 = 0$$

$$c_7 = \frac{c_5}{6} = -\frac{c_1}{2\cdot4\cdot6} ,$$

which yields a second linearly independent solution

$$y_2(x) = c_1 x - \frac{c_1}{2} x^3 + \frac{c_1}{2\cdot4} x^5$$

$$= c_1\left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot4} - \frac{x^7}{2\cdot4\cdot6} + \ldots\right) .$$

The general solution of equation (1.2) is

$$y(x) = c_0\left(1 - x^2 + \frac{x^4}{3} - \frac{x^6}{3\cdot5} + \ldots\right) \qquad (1.12)$$

$$+ c_1\left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot4} - \frac{x^7}{2\cdot4\cdot6} + \ldots\right) .$$