MATRIX ANALYSIS OF STRUCTURES

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1817
HARPER & ROW, PUBLISHERS, New York
Cambridge, Philadelphia, San Francisco,
London, Mexico City, São Paulo, Sydney

Sponsoring Editor: Cliff Robichaud
Project Editor: Cynthia L. Indriso
Production: Delia Tedoff
Compositor: Science Press
Printer and Binder: The Maple Press Company
Art Studio: Vantage Art, Inc.

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Library of Congress Cataloging in Publication Data

Meyers, V. James, 1933–
Matrix analysis of structures.

Includes index.
1. Structures, Theory of—Matrix methods. I. Title.
TA646.M49 1983 624.1'71 82-21308
ISBN 0-06-044438-X

NOTATION

a — distance from ith end of member to hinge location

$\mathbf{a}$ — array of force influence coefficients relating $\mathbf{R}$ and $\overline{\mathbf{P}}$

$\mathbf{a}^T$ — array of displacement influence coefficients relating $\boldsymbol{\Delta}$ and $\mathbf{D}$

$\mathbf{a}_R$ — subarray of $\mathbf{a}$ relating $\mathbf{R}$ and $\overline{\mathbf{P}}_R$

$\mathbf{a}_Q$ — subarray of $\mathbf{a}$ relating $\mathbf{R}$ and $\overline{\mathbf{P}}_Q$

A — member cross-sectional area

$\mathbf{A}$ — array of nodal equilibrium coefficients relating $\mathbf{R}$ and $\bar{\mathbf{r}}$

$\mathbf{A}^T$ — array of nodal compatibility coefficients relating $\bar{\mathbf{d}}$ and $\mathbf{D}$

b — distance from jth end of member to hinge location

$\mathbf{b}$ — array of force influence coefficients relating $\overline{\mathbf{P}}$ and $\mathbf{R}$ for statically determinate structures or $\overline{\mathbf{P}}$ and $\langle \mathbf{R} \mid \mathbf{Q} \rangle$ for statically indeterminate structures; thus $\mathbf{b} = \mathbf{a}^{-1}$ for statically determinate structures

$\mathbf{b}^T$ — array of displacement influence coefficients relating $\mathbf{D}$ and $\overline{\boldsymbol{\Delta}}$ for statically determinate structures or relating $\langle \mathbf{D} \mid \mathbf{D}_Q \rangle$ and $\overline{\boldsymbol{\Delta}}$ for statically indeterminate structures; thus $\mathbf{b}^T = [\mathbf{a}^T]^{-1}$ for statically determinate structures

$\mathbf{b}_R$ — subarray of $\mathbf{b}$ relating $\overline{\mathbf{P}}$ and $\mathbf{R}$

$\mathbf{b}_R^T$ — subarray of $\mathbf{b}^T$ relating $\mathbf{D}$ and $\overline{\boldsymbol{\Delta}}$

$\mathbf{b}_Q$ — subarray of $\mathbf{b}$ relating $\overline{\mathbf{P}}$ and $\mathbf{Q}$

$\mathbf{b}_Q^T$ — subarray of $\mathbf{b}^T$ relating $\mathbf{D}_Q$ and $\overline{\boldsymbol{\Delta}}$

c — cosine α

c_i, c_j — cosine α_i and cosine α_j, respectively

$\mathbf{d}$ — vector of member-end displacements, global coordinates

$\mathbf{d}'$ — vector of member-end displacements, local coordinates

$\bar{\mathbf{d}}$ — vector of member-end displacements for the collection of all members and supports, global coordinates

$\hat{\mathbf{d}}$ — vector of member-end displacements, skew coordinates

$\mathbf{d}_0$ — vector of member-end displacements due to initial strains, global coordinates

d_{ij} — element of $\mathcal{D}$ array

$\mathbf{d}_E$, $\mathbf{d}_I$ — vector of substructure displacements associated with external and internal degrees of freedom, respectively

$\mathbf{d}_c$ — vector of support-end displacements, global coordinates

$\mathcal{D}$ — diagonal coefficient array generated during modified gaussian decomposition

$\mathbf{D}$ — vector of structure nodal displacements, global coordinates

$\hat{\mathbf{D}}$ — vector of structure nodal displacements, generalized coordinates or mixed coordinates

$\mathbf{D}_0$	vector of structure nodal displacements due to initial strains, global coordinates
$\mathbf{D}_A$, $\mathbf{D}_S$	respective antisymmetric and symmetric components of the **D** vector for a symmetric structure
$\mathbf{D}_c$, $\mathbf{D}_u$	subarrays of the **D** vector associated with constrained and unconstrained degrees of freedom, respectively
$\mathbf{D}_Q$	vector of relative displacements corresponding to the vector of redundant forces, **Q**
$\mathbf{D}_R$, $\mathbf{D}_L$	subarrays of the **D** vector for the right and left sides, respectively, of a symmetric structure
E	modulus of elasticity
$\mathbf{E}$	elementary matrix
$\boldsymbol{f}$	array of member basic flexibility coefficients
$\overline{\boldsymbol{f}}$	array of member basic flexibility coefficients for the collection of all members and supports
f_c	constraint basic flexibility coefficient
$\mathbf{F}$	array of structure flexibility coefficients, global coordinates
$\mathbf{F}_{RR}$	subarray of the **F** matrix relating **D** and **R**
$\mathbf{F}_{RQ}$	subarray of the **F** matrix relating **D** and **Q**
$\mathbf{F}_{QR}$	subarray of the **F** matrix relating $\mathbf{D}_Q$ and **R**
$\mathbf{F}_{QQ}$	subarray of the **F** matrix relating $\mathbf{D}_Q$ and **Q**
g_{ij}	element of $\mathcal{G}$ array
$\mathcal{G}$	upper triangular array generated during gaussian decomposition
G	modulus of rigidity
h	depth of flexural member
i	end of member where $x' = 0$
I	moment of inertia of member cross section
$\mathbf{I}$	identity matrix
j	end of member where $x' = L$
J	member torsional geometric constant
$\boldsymbol{k}$	array of member basic stiffness coefficients
$\mathbf{k}$	array of member stiffness coefficients, global coordinates
$\mathbf{k}'$	array of member stiffness coefficients, local coordinates
$\hat{\mathbf{k}}$	array of member stiffness coefficients, skew coordinates
$\overline{\mathbf{k}}$	array of stiffness coefficients for collection of all members and supports, global coordinates
$\mathbf{k}_{EE}$	substructure stiffness array relating $\mathbf{r}_E$ and $\mathbf{d}_E$
$\mathbf{k}_{EI}$	substructure stiffness array relating $\mathbf{r}_E$ and $\mathbf{d}_I$

(*Continued on next page*)

$\mathbf{k}_{IE}$	substructure stiffness array relating $\mathbf{r}_I$ and $\mathbf{d}_E$
$\mathbf{k}_{II}$	substructure stiffness array relating $\mathbf{r}_I$ and $\mathbf{d}_I$
k_c	constraint basic stiffness coefficient
$\mathbf{k}_c$	constraint stiffness coefficient, global coordinates
$\mathbf{K}$	array of structure stiffness coefficients, global coordinates
$\hat{\mathbf{K}}$	array of structure stiffness coefficients, generalized coordinates or mixed coordinates
$\mathbf{K}_A$, $\mathbf{K}_S$	respective antisymmetric and symmetric components of the **K** matrix for a symmetric structure
$\mathbf{K}_{cc}$	subarray of the **K** matrix relating $\mathbf{R}_c$ and $\mathbf{D}_c$
$\mathbf{K}_{cu}$	subarray of the **K** matrix relating $\mathbf{R}_c$ and $\mathbf{D}_u$
$\mathbf{K}_{uc}$	subarray of the **K** matrix relating $\mathbf{R}_u$ and $\mathbf{D}_c$
$\mathbf{K}_{uu}$	subarray of the **K** matrix relating $\mathbf{R}_u$ and $\mathbf{D}_u$
$\mathbf{K}_{LL}$	subarray of the **K** matrix relating $\mathbf{R}_L$ and $\mathbf{D}_L$
$\mathbf{K}_{LR}$	subarray of the **K** matrix relating $\mathbf{R}_L$ and $\mathbf{D}_R$
$\mathbf{K}_{RL}$	subarray of the **K** matrix relating $\mathbf{R}_R$ and $\mathbf{D}_L$
$\mathbf{K}_{RR}$	subarray of the **K** matrix relating $\mathbf{R}_R$ and $\mathbf{D}_R$
ℓ_1, ℓ_2, ℓ_3	direction cosines of angles α_x, α_y, and α_z, respectively
ℓ_{ij}	element of $\mathcal{L}$ and $\mathcal{L}^T$
$\bar{\ell}_{ij}$	element of $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}^T$
$\mathcal{L}$	upper triangular coefficient array generated during modified gaussian decomposition
$\mathcal{L}^T$	lower triangular coefficient array generated during gaussian decomposition and modified gaussian decomposition
$\overline{\mathcal{L}}$	upper triangular coefficient array generated during Choleskian decomposition
$\overline{\mathcal{L}}^T$	lower triangular coefficient array generated during Choleskian decomposition
L	member length
$\mathbf{L}$	array of member force influence coefficients relating $\mathbf{r}'$ and $\mathbf{P}$
$\mathbf{L}^T$	array of member displacement influence coefficients relating $\boldsymbol{\Delta}$ and $\mathbf{d}'$
$m1$, $m2$, $m3$	direction cosines of y' axis relative to x, y, and z axes, respectively
m_B	semi-bandwidth of structure stiffness array
M	member internal moment
M_i	member-end basic moment, end i
M_j	member-end basic moment, end j
M_c	rotational constraint basic moment

(*Continued on back endpapers*)

Matrix Analysis of Structures

To L.R.M.

Preface

There is a growing awareness throughout society of the inexorable rise of the computer. For engineers, the computer is rapidly approaching a position of dominance with respect to their technical activities. Thus, to be competitive, engineers must be capable of communicating fluently with the computer. For structural engineers, this requires an understanding of matrix structural analysis. Before studying matrix structural analysis as presented in this book, it would be desirable for students to have taken an introductory structural analysis course taught from either a matrix or a classical point of view. (Some previous study of matrix algebra would, of course, also be useful.) For course use, this book is written at a level suitable for third- or fourth-year undergraduate students or beginning graduate students. Although a classical structural analysis background is unnecessary to the understanding of the subjects presented, appropriate comparisons to classical methods are occasionally made for the benefit of students who may have such a background.

In this book, the developments of concepts are based firmly on physical reasoning. Readers are repeatedly guided to an appreciation for the physical meaning that underlies the matrix and coding operations presented. Even the matrix algebra review is offered in the context of a very idealized simple structural system rather than as a sterile set of mathematical rules.

With regard to the arrangement of subject matter, the concepts of equilibrium and compatibility are introduced first, allowing for the solution of statically determinate structures only. The range of problems that can be solved is then extended by developing stiffness and flexibility relationships. The displacement method of analysis is emphasized, with ultimate development of the direct stiffness method for automatic computation. The force method of analysis is also presented for use in an automatic computation environment. Special topics such as initial strain loadings, members of variable cross section, and shear deformation effects are relegated to a separate chapter so that the development of major ideas is uninterrupted. Thus an overall view of the displacement method is presented before adjustments for important special conditions are attempted. In line with this approach, subjects are discussed in a two-dimensional context throughout the book, with extension to three dimensions being reserved as a topic for the final chapter. Subjects related to the main theme that receive particularly effective treatment in this book are the often neglected topic of symmetry and the equally important topic of solutions of large sets of simultaneous equations.

Throughout the book, each discussion proceeds from the specific to the general. Most concepts are presented first in the context of a specific structural example and then generalized. Furthermore, the examples used are kept as simple as possible while still retaining the capacity to demonstrate the concept under discussion. These examples are reused throughout the book so that subtle differences in solution techniques are not obscured by gross differences in structures.

This book makes rather extensive use of the computer program SMIS3, which is an educational program, not a production program. Although users are relieved of much of the tedium of structural analysis calculations, they are forced to provide instructions to the computer regarding the major steps in the analysis they wish to accomplish. Thus the computer program enhances rather than obscures the readers' understanding of the analysis process. One especially useful feature of the SMIS3 program is that subroutines for certain of the important analysis steps have been written in such a way that their replacement is very convenient. Thus as the development of the subject proceeds, readers can write their own computer subroutines for particular analysis steps. The readers can thus gain the experience of transforming a structural analysis concept into FORTRAN logic without being burdened with input/output and other programming details that would otherwise consume so much of their time and effort. Proceeding in this fashion, the readers will not only learn the analysis concepts but will also have the opportunity to implement them on the computer. In the end, they will have written subroutines typical of each important step of the analysis, which should give them a reasonable insight into the functioning of a structural analysis program. Although the purpose of using the SMIS3 program is to promote a thorough understanding of matrix-computer analysis of structures, readers are also encouraged in the book to retain a healthy skepticism as to the correctness of computer results. Hand checks on computer-generated results are frequently suggested.

Because the major steps in most examples are displayed in detail independently of SMIS3, this book can also be effectively used with any other appropriate structural analysis program. It can also be used in case no computer program is available. However, it is certainly desirable for readers to be able to interact with the computer during the study of this subject if at all possible. The SMIS3 program is available at nominal reproduction, handling, and shipping cost from the author.

ACKNOWLEDGMENTS

The contributions of the many people who have helped in this effort are gratefully acknowledged. In particular, thanks are due to Dr. E. L. Wilson and Dr. A. F. Schkade for their respective permissions to use SMIS and SMIS2 as bases for SMIS3. The transformation of SMIS2 into SMIS3 is the product of the FORTRAN coding expertise as well as the creativity of thought of Messrs. William Hendrickson and Jamshid Gharib. Expert typing has been patiently supplied by Ms. Cheryl Haines, Ms. Rona Ginsburg, Ms. Molly Harrington, and Ms. Lucy Hohn. Special thanks are due to the reviewers for their helpful suggestions. The interest and encouragement shown by staff members of the School of Civil Engineering at Purdue University are sincerely appreciated. Thanks to Dr. R. H. Lee for his help in the development of the SMIS3 program and to Professor A. D. M. Lewis for permission to use his unformatted input subroutine in SMIS3. The extraordinary efforts of Dr. Atef Saleeb in proofreading the manuscript and in meticulously working out examples for the instructor's manual for this text deserve very special mention. Lastly but most importantly, thanks to my wife, Lavon, for her understanding and support.

V. JAMES MEYERS

Contents

Chapter 1
Concepts of Structural Analysis

1.1 INTRODUCTION

Structures can be classified in a variety of ways. The casual observer might first consider classifying structures according to their respective functions: buildings, bridges, ships, aircraft, towers, and so on. This basis for structural classification is in fact fundamental; all structures have some functional reasons for existence. It is the need to fulfill some function that prompts the designer to give life to a structure. Furthermore, it is the need for a safe, serviceable, feasible, and aesthetically pleasing fulfillment of a function that dictates the form, material, and manner of loading of a structure.

Once the form and material have been determined, a structure may be further classified according to either its form (e.g., an arch, truss, or suspension structure) or the material out of which it is constructed (e.g., steel, concrete, or timber). The form and material of a structure in turn dictate its behavior, which in turn dictates the character of the analytical model. Figure 1.1 illustrates schematically the relationships among the function a structure is to fulfill, the form and material of and loading on the structure, the behavior of the structure, and the analytical model of the structure.

At this point we need to discuss some of the aspects of structural behavior indicated in Fig. 1.1 and to explain their respective relationships to the form and material of the structure. A structure is linear if its response to loading, say

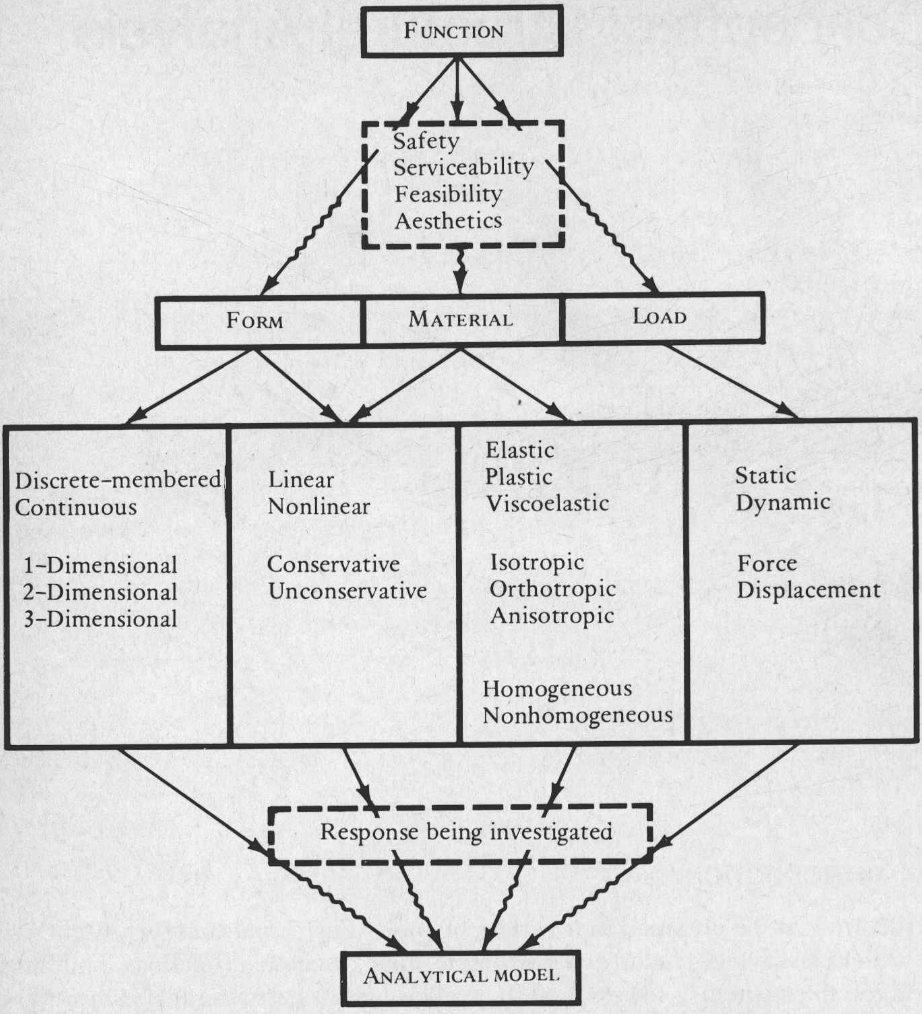

Fig. 1.1 Relationship between the function of a structure and its analytical model.

displacement at a point, is directly proportional to the magnitude of the applied load. If this proportionality does not exist, the structure is said to be nonlinear. Structural nonlinearities are of two types: (1) material nonlinearities that arise when stress is not proportional to strain, and (2) geometric nonlinearities that arise when the configuration of the structure under load is markedly changed from the unloaded configuration. (The presence of cables in a structure often leads to geometric nonlinearity because displacements can occur owing to a change in cable sag, which can be shown to be nonlinearly related to the force in the cable.) Materials, and therefore structures built from them, may be classified as elastic, plastic, or viscoelastic. Elastic materials rebound to their initial configuration when the load is removed, whereas plastic materials retain a permanent set. The deformations of viscoelastic materials depend on time and therefore load history, whereas the deformations of elastic and plastic materials do not. A structural system is unconservative or conservative depending on whether or not energy is lost from the system during a cycle of loading and unloading. Energy is generally lost if a system does not recover its initial shape after unloading owing either to plastic behavior of the material or to friction forces within or between parts of the structure.

All these behavioral aspects of the structure will have a significant influence on the nature of the analysis used in studying the structure. In addition, in developing the analytical model it will be necessary to consider whether the structural material is homogeneous or nonhomogeneous and whether it is isotropic, orthotropic, or anisotropic. (The physical properties of homogeneous materials are the same at each point; those of nonhomogeneous materials are not. The physical properties of isotropic materials are the same in all directions at a point; those of anisotropic materials are not. An orthotropic material is a special anisotropic material whose properties are different in three principal directions but whose properties in all other directions are dependent on those in the principal directions.) Other aspects of the structure, although important design considerations, will not usually have a significant impact on the analysis technique. These include brittleness, ductility, flammability, texture, color, hardness, and machinability.

Finally, the nature of the loading, which is dependent on the function of the structure, will also influence the analysis. The only truly static loading on a structure is the dead, or gravity, loading. However, if other loadings are applied gradually enough, they are called quasi-static loadings and may be considered static for analysis purposes. Whether or not the rate of loading is gradual enough depends on whether or not the time it takes to apply the load is longer than the fundamental period of vibration of the structure being analyzed. Loads usually need be treated as dynamic only if they are periodic in nature or if they are applied very suddenly. Even then, sometimes an "impact factor" is applied to an analysis with a static-loading result to account for the effect of a suddenly applied load. Loads can also be categorized as either external applied forces or internal initial distortions. Thermal loading is an example of an internal initial distortion (or initial strain) loading. (It will be shown in Chap. 2 that initial external displacements can also be considered to be initial internal distortions in support members.)

Unfortunately, the picture of structural behavior is generally not so clear as that just painted. That is, materials are not either "linear" or "nonlinear" and "elastic" or "plastic"; instead, their behavior depends on circumstances such as environment and rate of loading. The picture is further clouded in that the type of behavior that must be considered in an analysis may depend on the type of response being investigated. For example, a simpler analytical model may suffice to obtain static displacement and stress results than that which would be required for vibration or buckling results.

To clarify this picture for purposes of a rational presentation of matrix analysis of structures, we will make simplifying assumptions as to the nature of the behavior of structures. Thus we will consider only the displacement and stress response due to static loading of linear, elastic, conservative structures. We will further restrict our attention to discrete-membered structures (rigid- and pin-jointed frameworks) as opposed to continuous structures. However, it is important to recognize at the outset that the concepts that will be presented can be extended to the solution of many other classes of structural problems, including those involving dynamic response, material and geometric nonlinearities, inelasticity, instability, and continuous systems. Furthermore, the same concepts can be applied to problems from other areas of engineering, such as geotechnics, hydraulics, and heat transfer, as well as to problems outside of engineering altogether. Finally, to conserve space and time, most of our studies will deal with planar structures subjected to planar loadings in the plane of the structure. This approach will retain enough generality that the resulting analysis methods can be readily extended to three-dimensional applications.

For the sake of brevity, we will refer to our subject as "matrix structural analysis." A major feature that will be evident in matrix structural analysis is an emphasis on a *systematic* approach to the statement of the problem. Matrix notation turns out to be convenient to use in this connection because of its shorthand characteristics. Furthermore, the systematic approach together with matrix notation makes it particularly convenient to translate the statement of the problem to a computer language. Our effort will therefore be to develop matrix structural analysis as a tool for automatic computation of structural response to load.

1.2 SOLUTION METHODS

As mentioned before, the aspects of structural response to static load with which we will be mainly concerned are the internal force (and therefore stress) distribution in the structure and the displacements that the structure undergoes. These force and displacement responses constitute two infinite sets of unknowns, some portions of which must be determined in order to design the structure. Questions naturally arise as to whether we should attempt to determine displacement or force unknowns first and also whether some finite-sized subsets of each of these sets exist that are in some way fundamental. It will be shown that the consideration of either displacements or forces as the primary unknowns can lead to an effective solution method. It will be further shown that it is possible to identify a basic set of forces associated with each member (Ref. 1.1), in that not only are these forces independent of one another, but also all other forces in that member are directly dependent on this set. Thus this set of forces constitutes the minimum set that is capable of completely defining the stressed state of the member. Similarly, it will be shown that there is a corresponding basic set of displacements associated with each member (Ref. 1.1), in that not only are these displacements independent of one another, but also they are just sufficient to completely define the distorted shape of the member.

1.2.1 Force Method of Analysis

If internal forces are selected as the primary unknowns in a structure, the analysis method is referred to as the force method. It is convenient to consider the force method in the context of a pin-jointed truss structure.