COMPLEX ANALYSIS

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Foreword

The present book is meant as a text for a course on complex analysis at the advanced undergraduate level, or first-year graduate level. Somewhat more material has been included than can be covered at leisure in one term, to give opportunities for the instructor to exercise his taste, and lead the course in whatever direction strikes his fancy at the time. A large number of routine exercises are included for the more standard portions, and a few harder exercises of striking theoretical interest are also included, but may be omitted in courses addressed to less advanced students.

In some sense, I think the classical German prewar texts were the best (Hurwitz-Courant, Knopp, Bieberbach, etc.) and I would recommend to anyone to look through them. More recent texts have emphasized connections with real analysis, which is important, but at the cost of exhibiting succinctly and clearly what is peculiar about complex analysis: the power series expansion, the uniqueness of analytic continuation, and the calculus of residues. The systematic elementary development of formal and convergent power series was standard fare in the German texts, but only Cartan, in the more recent books, includes this material, which I think is quite essential, e.g. for differential equations. I have written a short text, exhibiting these features, making it applicable to a wide variety of tastes.

The book essentially decomposes in two parts.

The *first part*, Chapters I through VIII, includes the basic properties of analytic functions, essentially what cannot be left out of, say, a one-semester course.

I have no fixed idea about the manner in which Cauchy's theorem is to be treated. In less advanced classes, or if time is lacking, the usual hand waving about simple closed curves and interiors is not entirely inappropriate. Perhaps better would be to state precisely the homological version and omit the formal proof. For those who want a more thorough understanding, I include the relevant material.

Artin originally had the idea of basing the homology needed for complex variables on the winding number. I have included his proof for Cauchy's theorem, extracting, however, a purely topological lemma of independent interest, not made explicit in Artin's original *Notre Dame* notes (cf. collected works) or in Ahlfors' book closely following Artin. I have also included the more recent proof by Dixon, which uses the winding number, but replaces the topological lemma by greater use of elementary properties of analytic functions which can be derived directly from the local theorem. The two aspects, homotopy and homology, enter both in an essential fashion for different applications of analytic functions, and neither is slighted at the expense of the other.

Most expositions usually include some of the global geometric properties of analytic maps at an early stage. I chose to make the preliminaries on complex functions as short as possible to get quickly into the analytic part of complex function theory: power series expansions and Cauchy's theorem. The advantages of doing this, reaching the heart of the subject rapidly, are obvious. The cost is that certain elementary global geometric considerations are thus omitted from Chapter I, for instance, to reappear later in connection with analytic isomorphisms (Conformal mappings, Chapter VII) and potential theory (Harmonic functions, Chapter VIII). I think it is best for the coherence of the book to have covered in one sweep the basic analytic material before dealing with these more geometric global topics. Since the proof of the general Riemann mapping theorem is somewhat more difficult than the study of the specific cases considered in Chapter VII, it has been postponed to the second part.

This *second part* of the book, Chapters IX through XIV, deals with further assorted analytic aspects of functions in many directions, which may lead to many other branches of analysis. I have emphasized the possibility of defining analytic functions by an integral involving a parameter and differentiating under the integral sign. Some classical functions, of Bessel and Whittaker type, are given to work out as exercises, but the gamma function is worked out in detail in the text, as a prototype. *The chapters in this part are essentially logically independent and can be covered in any order, or omitted at will.*

In particular, the chapter on analytic continuation, including the Schwarz reflection principle, and/or the proof of the Riemann mapping theorem could be done right after Chapter VII, and still achieve great coherence.

As most of this part is somewhat harder than the first part, it can easily be omitted from a course addressed to undergraduates. In the same spirit, some of the harder exercises in the first part have been starred, to make their omission easy.

New Haven, Connecticut S.L.
October 1976

Prerequisites

We assume that the reader has had two years of calculus, and has some acquaintance with epsilon–delta techniques. For his convenience, we have recalled all the necessary lemmas we need for continuous functions on compact sets in the plane.

We use what is now standard terminology. A function

$$f: S \longrightarrow T$$

is called **injective** if $x \neq y$ in S implies $f(x) \neq f(y)$. It is called **surjective** if for every z in T there exists $x \in S$ such that $f(x) = z$. If f is surjective, then we also say that f maps S **onto** T. If f is both injective and surjective then we say that f is **bijective**.

Given two functions f, g defined on a set of real numbers containing arbitrarily large numbers, and such that $g(x) \geqslant 0$, we write

$$f \ll g \qquad \text{or} \qquad f(x) \ll g(x) \qquad \text{for} \quad x \longrightarrow \infty$$

to mean that there exists a number $C > 0$ such that for all x sufficiently large, we have

$$|f(x)| \leqslant Cg(x).$$

Similarly, if the functions are defined for x near 0, we use the same symbol $\ll$ for $x \longrightarrow 0$ to mean that there

$$|f(x)| \leqslant Cg(x)$$

for all x sufficiently small (there exists $\delta > 0$ such that if $|x| < \delta$ then $|f(x)| \leqslant Cg(x)$). Often this relation is also expressed by writing

$$f(x) = O(g(x)),$$

which is read: $f(x)$ is **big oh of** $g(x)$, for $x \longrightarrow \infty$ or $x \longrightarrow 0$ as the case may be.

We use $]a, b[$ to denote the **open** interval of numbers

$$a < x < b.$$

Similarly, $[a, b[$ denotes the half-open interval, etc.

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BASIC THEORY

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Basic Theory

I Complex Numbers and Functions

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in real numbers. For instance, $x^2 = 2$ is such an equation. However, we also know some equations having no solution in real numbers, for instance $x^2 = -1$, or $x^2 = -2$. We define a new kind of number where such equations have solutions. The new kind of numbers will be called **complex** numbers.

§1. DEFINITION

The **complex numbers** are a set of objects which can be added and multiplied, the sum and product of two complex numbers being also a complex number, and satisfy the following conditions.

1. Every real number is a complex number, and if α, β are real numbers, then their sum and product as complex numbers are the same as their sum and product as real numbers.
2. There is a complex number denoted by i such that $i^2 = -1$.
3. Every complex number can be written uniquely in the form $a + bi$ where a, b are real numbers.
4. The ordinary laws of arithmetic concerning addition and multiplication are satisfied. We list these laws:

   If α, β, γ are complex numbers, then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, and

   $$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

   We have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, and $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$.

   We have $\alpha\beta = \beta\alpha$, and $\alpha + \beta = \beta + \alpha$.

   If 1 is the real number one, then $1\alpha = \alpha$.

   If 0 is the real number zero, then $0\alpha = 0$.

   We have $\alpha + (-1)\alpha = 0$.

We shall now draw consequences of these properties. With each complex number $a + bi$, we associate the point (a, b) in the plane. Let $\alpha = a_1 + a_2 i$ and $\beta = b_1 + b_2 i$ be two complex numbers. Then

$$\alpha + \beta = a_1 + b_1 + (a_2 + b_2)i.$$

Hence addition of complex numbers is carried out "componentwise". For example, $(2 + 3i) + (-1 + 5i) = 1 + 8i$.

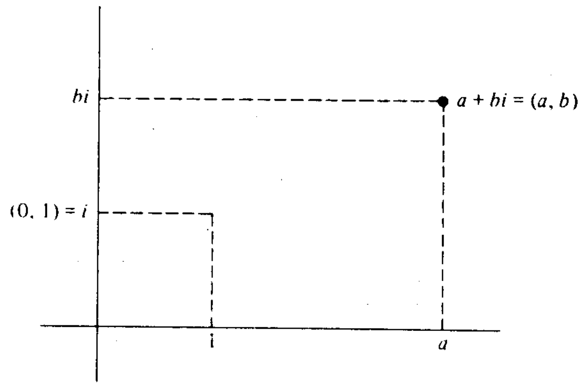

Figure 1

In multiplying complex numbers, we use the rule $i^2 = -1$ to simplify a product and to put it in the form $a + bi$. For instance, let $\alpha = 2 + 3i$ and $\beta = 1 - i$. Then

$$\begin{aligned} \alpha\beta = (2 + 3i)(1 - i) &= 2(1 - i) + 3i(1 - i) \\ &= 2 - 2i + 3i - 3i^2 \\ &= 2 + i - 3(-1) \\ &= 2 + 3 + i \\ &= 5 + i. \end{aligned}$$

Let $\alpha = a + bi$ be a complex number. We define $\bar{\alpha}$ to be $a - bi$. Thus if $\alpha = 2 + 3i$, then $\bar{\alpha} = 2 - 3i$. The complex number $\bar{\alpha}$ is called the **conjugate** of α. We see at once that

$$\alpha\bar{\alpha} = a^2 + b^2.$$

With the vector interpretation of complex numbers, we see that $\alpha\bar{\alpha}$ is the square of the distance of the point (a, b) from the origin.

We now have one more important property of complex numbers, which will allow us to divide by complex numbers other than 0.

If $\alpha = a + bi$ is a complex number $\neq 0$, and if we let

$$\lambda = \frac{\bar{\alpha}}{a^2 + b^2}$$

then $\alpha\lambda = \lambda\alpha = 1$.

The proof of this property is an immediate consequence of the law of multiplication of complex numbers, because

$$\alpha \frac{\bar{\alpha}}{a^2 + b^2} = \frac{\alpha\bar{\alpha}}{a^2 + b^2} = 1.$$

The number λ above is called the **inverse** of α, and is denoted by α^{-1} or $1/\alpha$. If α, β are complex numbers, we often write β/α instead of $\alpha^{-1}\beta$ (or $\beta\alpha^{-1}$), just as we did with real numbers. We see that we can divide by complex numbers $\neq 0$.

Example. To find the inverse of $(1 + i)$ we note that the conjugate of $1 + i$ is $1 - i$ and that $(1 + i)(1 - i) = 2$. Hence

$$(1 + i)^{-1} = \frac{1 - i}{2}.$$

Theorem 1.1 *Let α, β be complex numbers. Then*

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}, \qquad \overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}, \qquad \bar{\bar{\alpha}} = \alpha.$$

Proof. The proofs follow immediately from the definitions of addition, multiplication, and the complex conjugate. We leave them as exercises (Exercises 3 and 4).

Let $\alpha = a + bi$ be a complex number, where a, b are real. We shall call a the **real part** of α, and denote it by $\mathrm{Re}(\alpha)$. Thus

$$\alpha + \bar{\alpha} = 2a = 2\,\mathrm{Re}(\alpha).$$

The real number b is called the **imaginary part** of α, and denoted by $\mathrm{Im}(\alpha)$.

We define the **absolute value** of a complex number $\alpha = a_1 + ia_2$ (where a_1, a_2 are real) to be

$$|\alpha| = \sqrt{a_1^2 + a_2^2}.$$

If we think of α as a point in the plane (a_1, a_2), then $|\alpha|$ is the length of the line segment from the origin to α. In terms of the absolute value, we can write

$$\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2}$$

provided $\alpha \neq 0$. Indeed, we observe that $|\alpha|^2 = \alpha\bar{\alpha}$.

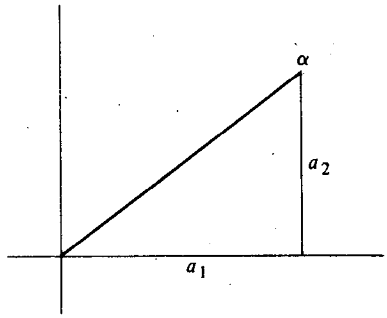

Figure 2

If $\alpha = a_1 + ia_2$, we note that

$$|\alpha| = |\bar{\alpha}|$$

because $(-a_2)^2 = a_2^2$, so $\sqrt{a_1^2 + a_2^2} = \sqrt{a_1^2 + (-a_2)^2}$.

Theorem 1.2 *The absolute value of a complex number satisfies the following properties. If α, β are complex numbers, then*

$$|\alpha\beta| = |\alpha|\,|\beta|$$
$$|\alpha + \beta| \leqslant |\alpha| + |\beta|.$$

Proof. We have

$$|\alpha\beta|^2 = \alpha\beta\,\overline{\alpha\beta} = \alpha\bar{\alpha}\beta\bar{\beta} = |\alpha|^2|\beta|^2.$$

Taking the square root, we conclude that $|\alpha|\,|\beta| = |\alpha\beta|$, thus proving the first assertion. As for the second, we have

$$\begin{aligned} |\alpha + \beta|^2 = (\alpha + \beta)(\overline{\alpha + \beta}) &= (\alpha + \beta)(\bar{\alpha} + \bar{\beta}) \\ &= \alpha\bar{\alpha} + \beta\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\beta} \\ &= |\alpha|^2 + 2\,\mathrm{Re}(\beta\bar{\alpha}) + |\beta|^2 \end{aligned}$$

because $\alpha\bar{\beta} = \overline{\beta\bar{\alpha}}$. However, we have

$$2\,\mathrm{Re}(\beta\bar{\alpha}) \leqslant 2|\beta\bar{\alpha}|$$

because the real part of a complex number is $\le$ its absolute value. Hence

$$\begin{aligned}|\alpha + \beta|^2 &\le |\alpha|^2 + 2|\beta\bar{\alpha}| + |\beta|^2 \\ &\le |\alpha|^2 + 2|\beta|\,|\alpha| + |\beta|^2 \\ &= (|\alpha| + |\beta|)^2.\end{aligned}$$

Taking the square root yields the second assertion of the theorem.

The inequality

$$|\alpha + \beta| \le |\alpha| + |\beta|$$

is called the **triangle inequality**. It also applies to a sum of several terms. If $z_1, \ldots, z_n$ are complex numbers then we have

$$|z_1 + \cdots + z_n| \le |z_1| + \cdots + |z_n|.$$

Also observe that for any complex number z, we have

$$|-z| = |z|.$$

Proof?

EXERCISES

1. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.
   (a) $(-1 + 3i)^{-1}$ (b) $(1 + i)(1 - i)$
   (c) $(1 + i)i(2 - i)$ (d) $(i - 1)(2 - i)$
   (e) $(7 + \pi i)(\pi + i)$ (f) $(2i + 1)\pi i$
   (g) $(\sqrt{2}\, i)(\pi + 3i)$ (h) $(i + 1)(i - 2)(i + 3)$
2. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.
   (a) $(1 + i)^{-1}$ (b) $\dfrac{1}{3 + i}$ (c) $\dfrac{2 + i}{2 - i}$ (d) $\dfrac{1}{2 - i}$
   (e) $\dfrac{1 + i}{i}$ (f) $\dfrac{i}{1 + i}$ (g) $\dfrac{2i}{3 - i}$ (h) $\dfrac{1}{-1 + i}$
3. Let α be a complex number $\neq 0$. What is the absolute value of $\alpha/\bar{\alpha}$? What is $\bar{\bar{\alpha}}$?
4. Let α, β be two complex numbers. Show that $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$ and that

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}.$$

5. Justify the assertion made in the proof of Theorem 2, that the real part of a complex number is $\le$ its absolute value.

6. If $\alpha = a + ib$ with a, b real, then b is called the **imaginary part** of α and we write $b = \mathrm{Im}(\alpha)$. Show that $\alpha - \bar{\alpha} = 2i\,\mathrm{Im}(\alpha)$. Show that

$$\mathrm{Im}(\alpha) \leq |\mathrm{Im}(\alpha)| \leq |\alpha|.$$

7. Find the real and imaginary parts of $(1 + i)^{100}$.
8. Prove that for any two complex numbers z, w we have:
   (a) $|z| \leq |z - w| + |w|$
   (b) $|z| - |w| \leq |z - w|$
   (c) $|z| - |w| \leq |z + w|$
9. Let $\alpha = a + ib$ and $z = x + iy$. Let c be real > 0. Transform the condition

$$|z - \alpha| = c$$

   into an equation involving only x, y, a, b and describe in a simple way what geometric figure is represented by this equation.
10. Describe the set of points z satisfying the following conditions geometrically.
   (a) $|z - i + 3| = 5$ (b) $|z - i + 3| > 5$
   (c) $|z - i + 3| \leq 5$ (d) $|z + 2i| < 1$
   (e) $\mathrm{Im}\, z > 0$ (f) $\mathrm{Im}\, z \geq 0$
   (g) $\mathrm{Re}\, z > 0$ (h) $\mathrm{Re}\, z \leq 0$
11. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be complex numbers. Assume that $a_1, \ldots, a_n$ are distinct. Find a polynomial $P(z)$ of degree at most $n - 1$ such that $P(a_j) = b_j$ for $j = 1, \ldots, n$. Prove that such a polynomial is unique. [*Hint*: Use the Vandermonde determinant.]

§2. POLAR FORM

Let $(x, y) = x + iy$ be a complex number. We know that any point in the plane can be represented by polar coordinates (r, θ). We shall now see how to write our complex number in terms of such polar coordinates.

Let θ be a real number. We define the expression $e^{i\theta}$ to be

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus $e^{i\theta}$ is a complex number.

For example, if $\theta = \pi$, then $e^{i\pi} = -1$. Also, $e^{2\pi i} = 1$, and $e^{i\pi/2} = i$. Furthermore, $e^{i(\theta + 2\pi)} = e^{i\theta}$ for any real θ.

Let x, y be real numbers and $x + iy$ a complex number. Let

$$r = \sqrt{x^2 + y^2}\,.$$

If (r, θ) are the polar coordinates of the point (x, y) in the plane, then

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$