GIAN-CARLO ROTA, *Editor*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Volume 2

Section: Number Theory

Paul Turán, *Section Editor*

The Theory of Partitions

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

The theory of partitions is one of the very few branches of mathematics that can be appreciated by anyone who is endowed with little more than a lively interest in the subject. Its applications are found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic studies of matter, in the theory of numbers, or in combinatorial problems from all sources.

Professor Andrews has written the first thorough survey of this many-sided field. The specialist will consult it for the more recondite results, the student will be challenged by many a deceptively simple fact, and the applied scientist may locate in it the missing identity he needs to organize his data.

Professor Turán's untimely death has left this book without a suitable introduction. It is fitting to dedicate it to the memory of one of the masters of number theory.

GIAN-CARLO ROTA

Preface

Let us begin by acknowledging that the word "partition" has numerous meanings in mathematics. Any time a division of some object into subobjects is undertaken, the word partition is likely to pop up. For the purposes of this book a "partition of n" is a nonincreasing finite sequence of positive integers whose sum is n. We shall extend this definition in Chapters 11, 12, and 13 when we consider higher-dimensional partitions, partitions of n-tuples, and partitions of sets, respectively. Compositions or ordered partitions (merely finite sequences of positive integers) will be considered in Chapter 4.

The theory of partitions has an interesting history. Certain special problems in partitions certainly date back to the Middle Ages; however, the first discoveries of any depth were made in the eighteenth century when L. Euler proved many beautiful and significant partition theorems. Euler indeed laid the foundations of the theory of partitions. Many of the other great mathematicians—Cayley, Gauss, Hardy, Jacobi, Lagrange, Legendre, Littlewood, Rademacher, Ramanujan, Schur, and Sylvester—have contributed to the development of the theory.

There have been almost no books devoted entirely to partitions. Generally the combinatorial and formal power series aspects of partitions have found a place in older books on elementary analysis (*Introductio in Analysin Infinitorum* by Euler, *Textbook of Algebra* by Chrystal), in encyclopedic surveys of number theory (*Niedere Zahlentheorie* by Bachman, *Introduction to the Theory of Numbers* by Hardy and Wright), and in combinatorial analysis books (*Combinatory Analysis* by MacMahon, *Introduction to Combinatorial Analysis* by Riordan, *Combinatorial Methods* by Percus, *Advanced Combinatorics* by Comtet). The asymptotic problems associated with partitions have, on the other hand, been treated in works on analytic or additive number theory (*Introduction to the Analytic Theory of Numbers* by Ayoub, *Modular Functions in Analytic Number Theory* by Knopp, *Topics from the Theory of Numbers* by Grosswald, *Additive Zahlentheorie* by Ostmann, *Topics in Analytic Number Theory* by Rademacher).

If one considers the applications of partitions in various branches of mathematics and statistics, one is struck by the interplay of combinatorial and asymptotic methods. We have tried to organize this book so that it adequately develops and interrelates both combinatorial and analytic methods.

Chapters 1–4 treat the elementary portions of the theory of partitions; of primary importance here is the use of generating functions.

Chapters 5 and 6 treat the asymptotic problems. Partition identities are dealt with in Chapters 7 through 9. Chapter 10 on partition function congruences returns to the analytic aspect of partitions. Chapters 11–13 treat several generalizations of partitions and Chapter 14 presents a brief discussion of the computational aspect of partitions.

There are three concluding sections of each chapter: A "Notes" section provides historical comment on the material covered; a "References" section provides a substantial but nonexhaustive list of relevant books and papers; and an "Examples" section provides statements of results not fully covered in the text. Those examples that occur with an asterisk are significant advances beyond the material presented in the text; the remainder form a reasonable set of exercises by which the reader may determine his grasp of the subject matter. References for the source of the examples occur in the related Notes section.

Many of the mathematical sciences have seen applications of partitions recently. Nonparametric statistics require restricted partitions like those in Chapter 3. Various permutation problems in probability and statistics are intimately linked with the Simon Newcomb problem of Chapter 4. Particle physics uses partition asymptotics and partition identities related to the work in Chapters 5–9. Group theory (through Young tableaux) is intimately connected with Chapter 12, and the relationship between partitions and combinatorial theory is explored in Chapter 13.

The material in this book has been developed over a period of years. My first acquaintance with partitions came from thrilling lectures delivered by my thesis adviser, the late Professor Hans Rademacher. Many of the topics herein have been presented in graduate courses at the Pennsylvania State University between 1964 and 1975, in seminars at MIT during the 1970–1971 academic year, at the University of Erlangen in the summer of 1975, and at the University of Wisconsin during the 1975–1976 academic year. I owe a great debt of gratitude to many people at these four universities. I wish to thank specially R. Askey, K. Baclawski, B. Berndt, and L. Carlitz, who contributed many valuable suggestions and comments during the preparation of this book.

Finally I thank my wife, Joy, who has throughout this project been both a help and an inspiration to me.

GEORGE E. ANDREWS

Contents

CHAPTER 1

The Elementary Theory of Partitions

1.1 Introduction

In this book we shall study in depth the fundamental additive decomposition process: the representation of positive integers by sums of other positive integers.

DEFINITION 1.1. A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition.

Many times the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be denoted by λ, and we shall write $\lambda \vdash n$ to denote "λ is a partition of n." Sometimes it is useful to use a notation that makes explicit the number of times that a particular integer occurs as a part. Thus if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$, we sometimes write

$$\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \cdots)$$

where exactly f_i of the λ_j are equal to i. Note now that $\sum_{i \geq 1} f_i i = n$.

Numerous types of partition problems will concern us in this book; however, among the most important and fundamental is the question of enumerating various sets of partitions.

DEFINITION 1.2. The partition function $p(n)$ is the number of partitions of n.

Remark. Obviously $p(n) = 0$ when n is negative. We shall set $p(0) = 1$ with the observation that the empty sequence forms the only partition of zero. The following list presents the next six values of $p(n)$ and tabulates the actual partitions.

$p(1) = 1$: $1 = (1)$;
$p(2) = 2$: $2 = (2)$, $1 + 1 = (1^2)$;
$p(3) = 3$: $3 = (3)$, $2 + 1 = (12)$, $1 + 1 + 1 = (1^3)$;
$p(4) = 5$: $4 = (4)$, $3 + 1 = (13)$, $2 + 2 = (2^2)$,
$2 + 1 + 1 = (1^2 2)$, $1 + 1 + 1 + 1 = (1^4)$;

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$p(5) = 7$: $5 = (5)$, $4 + 1 = (14)$, $3 + 2 = (23)$,
$3 + 1 + 1 = (1^3 3)$, $2 + 2 + 1 = (12^2)$,
$2 + 1 + 1 + 1 = (1^3 2)$, $1 + 1 + 1 + 1 + 1 = (1^5)$;
$p(6) = 11$: $6 = (6)$, $5 + 1 = (15)$, $4 + 2 = (24)$,
$4 + 1 + 1 = (1^2 4)$, $3 + 3 = (3^2)$, $3 + 2 + 1 = (123)$,
$3 + 1 + 1 + 1 = (1^3 3)$, $2 + 2 + 2 = (2^3)$,
$2 + 2 + 1 + 1 = (1^2 2^2)$, $2 + 1 + 1 + 1 + 1 = (1^4 2)$,
$1 + 1 + 1 + 1 + 1 + 1 = (1^6)$.

The partition function increases quite rapidly with n. For example, $p(10) = 42$, $p(20) = 627$, $p(50) = 204226$, $p(100) = 190569292$, and $p(200) = 3972999029388$.

Many times we are interested in problems in which our concern does not extend to all partitions of n but only to a particular subset of the partitions of n.

DEFINITION 1.3. Let $\mathcal{S}$ denote the set of all partitions.

DEFINITION 1.4. Let $p(S, n)$ denote the number of partitions of n that belong to a subset S of the set $\mathcal{S}$ of all partitions.

For example, we might consider $\mathcal{O}$ the set of all partitions with odd parts and $\mathcal{D}$ the set of all partitions with distinct parts. Below we tabulate partitions related to $\mathcal{O}$ and to $\mathcal{D}$.

$p(\mathcal{O}, 1) = 1$: $1 = (1)$,
$p(\mathcal{O}, 2) = 1$: $1 + 1 = (1^2)$,
$p(\mathcal{O}, 3) = 2$: $3 = (3)$, $1 + 1 + 1 = (1^3)$,
$p(\mathcal{O}, 4) = 2$: $3 + 1 = (13)$, $1 + 1 + 1 + 1 = (1^4)$,
$p(\mathcal{O}, 5) = 3$: $5 = (5)$, $3 + 1 + 1 = (1^2 3)$,
$1 + 1 + 1 + 1 + 1 = (1^5)$,
$p(\mathcal{O}, 6) = 4$: $5 + 1 = (15)$, $3 + 3 = (3^2)$,
$3 + 1 + 1 + 1 = (1^3 3)$,
$1 + 1 + 1 + 1 + 1 + 1 = (1^6)$,
$p(\mathcal{O}, 7) = 5$: $7 = (7)$, $5 + 1 + 1 = (1^2 5)$, $3 + 3 + 1 = (13^2)$,
$3 + 1 + 1 + 1 + 1 = (1^4 3)$,
$1 + 1 + 1 + 1 + 1 + 1 + 1 = (1^7)$.

$p(\mathcal{D}, 1) = 1$: $1 = (1)$,
$p(\mathcal{D}, 2) = 1$: $2 = (2)$,
$p(\mathcal{D}, 3) = 2$: $3 = (3)$, $2 + 1 = (12)$,
$p(\mathcal{D}, 4) = 2$: $4 = (4)$, $3 + 1 = (13)$,
$p(\mathcal{D}, 5) = 3$: $5 = (5)$, $4 + 1 = (14)$, $3 + 2 = (23)$,
$p(\mathcal{D}, 6) = 4$: $6 = (6)$, $5 + 1 = (15)$, $4 + 2 = (24)$,
$3 + 2 + 1 = (123)$,
$p(\mathcal{D}, 7) = 5$: $7 = (7)$, $6 + 1 = (16)$, $5 + 2 = (25)$,
$4 + 3 = (34)$, $4 + 2 + 1 = (124)$.

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We point out the rather curious fact that $p(\mathcal{O}, n) = p(\mathcal{D}, n)$ for $n \leqslant 7$, although there is little apparent relationship between the various partitions listed (see Corollary 1.2).

In this chapter, we shall present two of the most elemental tools for treating partitions: (1) infinite product generating functions; (2) graphical representation of partitions.

1.2 Infinite Product Generating Functions of One Variable

DEFINITION 1.5. The generating function $f(q)$ for the sequence $a_0, a_1, a_2, a_3, \ldots$ is the power series $f(q) = \sum_{n \geqslant 0} a_n q^n$.

Remark. For many of the problems we shall encounter, it suffices to consider $f(q)$ as a "formal power series" in q. With such an approach many of the manipulations of series and products in what follows may be justified almost trivially. On the other hand, much asymptotic work (see Chapter 6) requires that the generating functions be analytic functions of the complex variable q. In actual fact, both approaches have their special merits (recently, E. Bender (1974) has discussed the circumstances in which we may pass from one to the other). Generally we shall state our theorems on generating functions with explicit convergence conditions. For the most part we shall be dealing with absolutely convergent infinite series and infinite products; consequently, various rearrangements of series and interchanges of summation will be justified analytically from this simple fact.

DEFINITION 1.6. Let H be a set of positive integers. We let "H" denote the set of all partitions whose parts lie in H. Consequently, $p(\text{“}H\text{”}, n)$ is the number of partitions of n that have all their parts in H.

Thus if H_0 is the set of all odd positive integers, then "H_0" $= \mathcal{O}$.

$$p(\text{“}H_0\text{”}, n) = p(\mathcal{O}, n).$$

DEFINITION 1.7. Let H be a set of positive integers. We let "H"$(\leqslant d)$ denote the set of all partitions in which no part appears more than d times and each part is in H.

Thus if N is the set of all positive integers, then $p(\text{“}N\text{”}(\leqslant 1), n) = p(\mathcal{D}, n)$.

THEOREM 1.1. *Let H be a set of positive integers, and let*

$$f(q) = \sum_{n \geqslant 0} p(\text{“}H\text{”}, n) q^n, \tag{1.2.1}$$

$$f_d(q) = \sum_{n \geqslant 0} p(\text{“}H\text{”}(\leqslant d), n) q^n. \tag{1.2.2}$$

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Then for $|q| < 1$

$$f(q) = \prod_{n \in H} (1 - q^n)^{-1}, \tag{1.2.3}$$

$$f_d(q) = \prod_{n \in H} (1 + q^n + \cdots + q^{dn})$$

$$= \prod_{n \in H} (1 - q^{(d+1)n})(1 - q^n)^{-1}. \tag{1.2.4}$$

Remark. The equivalence of the two forms for $f_d(q)$ follows from the simple formula for the sum of a finite geometric series:

$$1 + x + x^2 + \cdots + x^r = \frac{1 - x^{r+1}}{1 - x}.$$

Proof. We shall proceed in a formal manner to prove (1.2.3) and (1.2.4); at the conclusion of our proof we shall sketch how to justify our steps analytically. Let us index the elements of H, so that $H = \{h_1, h_2, h_3, h_4, \ldots\}$. Then

$$\prod_{n \in H} (1 - q^n)^{-1} = \prod_{n \in H} (1 + q^n + q^{2n} + q^{3n} + \cdots)$$

$$= (1 + q^{h_1} + q^{2h_1} + q^{3h_1} + \cdots)$$

$$\times (1 + q^{h_2} + q^{2h_2} + q^{3h_2} + \cdots)$$

$$\times (1 + q^{h_3} + q^{2h_3} + q^{3h_3} + \cdots)$$

$$\cdots$$

$$= \sum_{a_1 \geqslant 0} \sum_{a_2 \geqslant 0} \sum_{a_3 \geqslant 0} \cdots q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots}$$

and we observe that the exponent of q is just the partition $(h_1^{a_1} h_2^{a_2} h_3^{a_3} \cdots)$. Hence q^N will occur in the foregoing summation once for each partition of n into parts taken from H. Therefore

$$\prod_{n \in H} (1 - q^n)^{-1} = \sum_{n \geqslant 0} p(\text{“}H\text{”}, n) q^n.$$

The proof of (1.2.4) is identical with that of (1.2.3) except that the infinite geometric series is replaced by the finite geometric series:

$$\prod_{n \in H} (1 + q^n + q^{2n} + \cdots + q^{dn})$$

$$= \sum_{d \geqslant a_1 \geqslant 0} \sum_{d \geqslant a_2 \geqslant 0} \sum_{d \geqslant a_3 \geqslant 0} \cdots q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots}$$

$$= \sum_{n \geqslant 0} p(\text{“}H\text{”}(\leqslant d), n) q^n.$$

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If we are to view the foregoing procedures as operations with convergent infinite products, then the multiplication of infinitely many series together requires some justification. The simplest procedure is to truncate the infinite product to $\prod_{i=1}^{n}(1-q^{h_i})^{-1}$. This truncated product will generate those partitions whose parts are among $h_1, h_2, \ldots, h_n$. The multiplication is now perfectly valid since only a finite number of absolutely convergent series are involved. Now assume q is real and $0 < q < 1$; then if $M = h_n$,

$$\sum_{j=0}^{M} p(\text{“}H\text{”}, j)q^j \leqslant \prod_{i=1}^{n}(1-q^{h_i})^{-1} \leqslant \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} < \infty.$$

Thus the sequence of partial sums $\sum_{j=0}^{M} p(\text{“}H\text{”}, j)q^j$ is a bounded increasing sequence and must therefore converge. On the other hand

$$\sum_{j=0}^{\infty} p(\text{“}H\text{”}, j)q^j \geqslant \prod_{i=1}^{n}(1-q^{h_i})^{-1} \to \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} \qquad \text{as} \quad n \to \infty.$$

Therefore

$$\sum_{j=0}^{\infty} p(\text{“}H\text{”}, j)q^j = \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} = \prod_{n\in H}(1-q^n)^{-1}.$$

Similar justification can be given for the proof of (1.2.4). ■

Corollary 1.2 (Euler). *$p(\mathscr{O}, n) = p(\mathscr{D}, n)$ for all n.*

Proof. By Theorem 1.1,

$$\sum_{n\geqslant 0} p(\mathscr{O}, n)q^n = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$$

and

$$\sum_{n\geqslant 0} p(\mathscr{D}, n)q^n = \prod_{n=1}^{\infty}(1+q^n).$$

Now

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}. \tag{1.2.5}$$

Hence

$$\sum_{n\geqslant 0} p(\mathscr{O}, n)q^n = \sum_{n\geqslant 0} p(\mathscr{D}, n)q^n,$$

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and since a power series expansion of a function is unique, we see that $p(\mathcal{O}, n) = p(\mathcal{D}, n)$ for all n. ■

COROLLARY 1.3 (Glaisher). *Let N_d denote the set of those positive integers not divisible by d. Then*

$$p(\text{“}N_{d+1}\text{”}, n) = p(\text{“}N\text{”}(\leqslant d), n)$$

for all n.

Proof. By Theorem 1.1,

$$\sum_{n \geqslant 0} p(\text{“}N\text{”}(\leqslant d), n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{(d+1)n})}{(1 - q^n)}$$

$$= \prod_{\substack{n=1 \\ (d+1) \nmid n}}^{\infty} \frac{1}{(1 - q^n)}$$

$$= \sum_{n \geqslant 0} p(\text{“}N_{d+1}\text{”}, n)q^n,$$

and the result follows as before. ■

There are numerous results of the type typified by Corollaries 1.2 and 1.3. We shall run into such results again in Chapters 7 and 8, where much deeper theorems of a similar nature will be discussed.

1.3 Graphical Representation of Partitions

Another effective elementary device for studying partitions is the graphical representation. To each partition λ is associated its *graphical representation* $\mathcal{G}_\lambda$ (or Ferrers graph), which formally is the set of points with integral coordinates (i, j) in the plane such that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $(i, j) \in \mathcal{G}_\lambda$ if and only if $0 \geqslant i \geqslant -n + 1$, $0 \leqslant j \leqslant \lambda_{|i|+1} - 1$. Rather than dwell on this formal definition, we shall, by means of a few examples, fully explain the graphical representation.

The graphical representation of the partition $8 + 6 + 6 + 5 + 1$ is

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. . . . . . . .
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The graphical representation of the partition $7 + 3 + 3 + 2 + 1 + 1$ is

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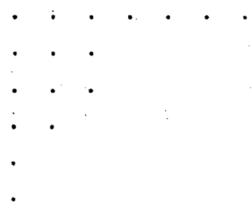
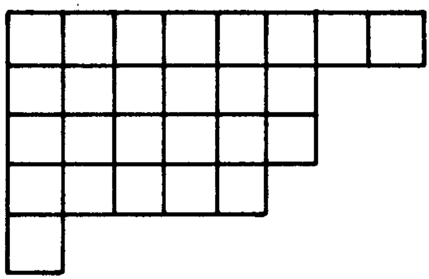


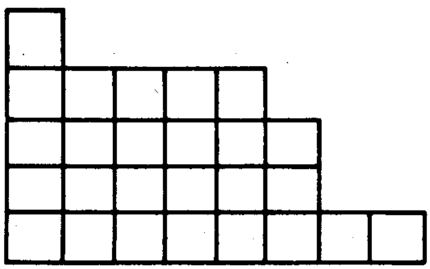
Note that the ith row of the graphical representation of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ contains λ_i points (or dots, or nodes).

We remark that there are several equivalent ways of forming the graphical representation. Some authors use unit squares instead of points, so that the graphical representation of $8 + 6 + 6 + 5 + 1$ becomes

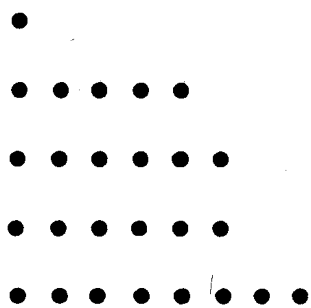


Such a representation is extremely useful when we consider applications of partitions to plane partitions or Young tableaux (see Chapter 11).

Other authors prefer the representation to be upside down (they would say right side up); for example, in the case of $8 + 6 + 6 + 5 + 1$



or



Since most of the classical texts on partitions use the first representation shown in this section, we shall also.

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Definition 1.8. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition, we may define a new partition $\lambda' = (\lambda_1', \ldots, \lambda_m')$ by choosing λ_i' as the number of parts of λ that are $\geqslant i$. The partition λ' is called the *conjugate* of λ.