

Mathematics Monograph Series **28**

Planar Dynamical Systems: Selected Classical Problems

Yirong Liu Jibin Li Wentao Huang
(刘一戎 李继彬 黄文韬)

(平面动力系统的若干经典问题)



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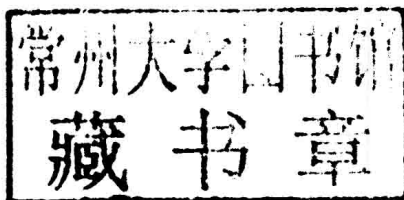
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Preface

In 2008, November 23-28, the workshop of “Classical Problems on Planar Polynomial Vector Fields ” were held in Banff International Research Station of Canada. So called “classical problems”, it concerns with the following: (1) Problems on integrability of planar polynomial vector fields. (2) The problem of the center stated by Poincaré for real polynomial differential systems, asks us to recognize when a planar vector field defined by polynomials of degree at most n possesses a singularity which is a center. (3) Global geometry of specific classes of planar polynomial vector fields. (4) Hilbert’s 16th problem.

These problems had been posed more than 110 years. Therefore, they are called “classical problem” in the studies of the theory of dynamical systems.

The qualitative theory and stability theory of differential equations, created by Poincaré and Lyapunov at the end of the 19th century had major developments as two branches of the theory of dynamical systems during the 20th century. As a part of the basic theory of nonlinear science, it is one of the very active areas in the new millennium.

This book presents in an elementary way the recent significant developments in the qualitative theory of planar dynamical systems. The subjects are covered as follows: The studies of center and isochronous center problems, multiple Hopf bifurcations and local and global bifurcations of the equivariant planar vector fields which concern with Hilbert’s 16th problem.

We are interested in the study of planar vector fields, because they occur very often in applications. Indeed, such equations appear in modelling chemical reactions, population dynamics, travelling wave systems of nonlinear evolution equations in mathematical physics and in many other areas of applied mathematics and mechanics. In the other hand, the study of planar vector fields has itself theoretical significance. We would like to cite Canada’s mathematician Dana Schlomiuk’s words to explain this fact: “Planar polynomial vector fields and more generally, algebraic differential equations over the projective space are interesting objects of study for their own sake. Indeed, due to their analytic, algebraic and geometric nature they form a fertile soil for intertwining diverse methods, and success in finding solutions to problems in this area depends very much on the capacity we have to blend the diverse aspects into a unified whole.”

We emphasize that for the problems of the planar vector fields, many sophisticated tools and theories have been built and still being developed, whose field of application goes far beyond the initial areas. In this book, we only state some important progress in the above directions which have attracted our study interest. The materials of this book are taken mainly from our published results.

This book is divided ten chapters. In Chapter 1 we provide some basic results in the theory of complex analytic autonomous systems. We discuss the normal forms,

integrability and linearized problem in a neighborhood of an elementary singular point.

In order to clearly understand the content in Chapter 2~Chapter 10 for young readers, and to save space in the following chapters, we shall describe in more detail the subjects which are written in this book and give brief survey of the historic literature.

I. Center-focus problem

We consider planar vector fields and their associated differential equations:

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \quad (E)$$

where $X(x, y), Y(x, y)$ are analytic functions or polynomials with real coefficients. If X, Y are polynomials, we call degree of a system (E) , the number $n = \max(\deg(X), \deg(Y))$. Without loss of generality, we assume that $X(0, 0) = Y(0, 0) = 0$, i.e., the origin $O(0, 0)$ is a singular point of (E) and the linearization at the origin of (E) has purely imaginary eigenvalues.

The origin $O(0, 0)$ is a *center* of (E) if there exists a neighborhood U of the origin such that every point in U other than $O(0, 0)$ is nonsingular and the orbit passing through the point is closed. In 1885, Poincaré posed the following problem.

The problem of the center Find necessary and sufficient conditions for a planar polynomial differential system (E) of degree m to possess a center.

This problem was solved in the case of quadratic systems by Dulac who proved that all quadratic systems with a center are integrable in finite terms. Actually they could be shown to be Darboux integrable by the method of Darboux by using invariant algebraic curves. Similar results were obtained for some classes of cubic differential systems with a center. Darboux integrability is an important tool, although not the only one. The problem of the center is open for general cubic systems and for higher degrees.

Poincaré considered the above problem. He gave an infinite set of necessary and sufficient conditions for such system to have a center at the origin. In his memoir on the stability of motion, Lyapunov studies systems of differential equations in n variables. When applied to the case $n = 2$, his results also gave an infinite set of necessary and sufficient conditions for system (E) with X, Y polynomials to have a center. (Actually, Lyapunov's result is more general since it is for the case where X and Y are analytic functions). In searching for sufficient conditions for a center, both Poincaré and Lyapunov's work involve the idea of trying to find a constant of the motion $F(x, y)$ for (E) in a neighborhood U of the origin, where

$$F(x, y) = \sum_{k=2}^{\infty} F_k(x, y), \quad (1)$$

F_k is a homogeneous polynomial of order k and F_2 is a positive definite quadratic form. If F is constant on all solution curve $(x(t), y(t))$ in U , we say that F is a first

integral on U of system (E) . If there exists such an F which is nonconstant on any open subset of U , we say that system (E) is integrable on U .

Poincaré and Lyapunov proved the following theorem.

Poincaré-Lyapunov Theorem The origin of the polynomial (or analytic) system (E) is a center if and only if in an open neighborhood U of the origin, (E) has a nonconstant first integral which is analytic.

Thus, we can construct a power series (1) such that

$$\left. \frac{dF}{dt} \right|_{(E)} = V_3(x^2 + y^2)^2 + V_5(x^2 + y^2)^3 + \cdots + V_{2k+1}(x^2 + y^2)^{k+1} + \cdots \quad (2)$$

with $V_3, V_5, \dots, V_{2k+1}, \dots$ constants. The first non-zero V_{2n+1} give the asymptotic stability or instability of the origin according to its negative or positive sign. Indeed, stopping the series at F_k , we obtain a polynomial which is a Lyapunov function for the system (E) . The V_{2k+1} 's are called the *Lyapunov constants*. Some people also use the term *focal values* for them. In fact, Andronov et al defined the focal values by the formula $\nu_i = \Delta^{(i)}(0)/i!$, where $\Delta^{(i)}(\rho_0)$ is the i th-derivative of the function $\Delta(\rho_0) = P(\rho_0) - \rho_0$, P is the Poincaré return map. The first non-zero focal value of Andronov corresponding to an odd number $i = 2n + 1$. It had been proved that the first non-zero Lyapunov constant V_{2n+1} differs only by a positive constant factor from the first non-zero focal value, which is $\Delta^{(2n+1)}(0)$. Hence, the identification in the terminology is natural.

In terms of the V_{2i+1} 's, the conditions for a center of the origin become $V_{2k+1} = 0$, for all $k = 1, 2, 3, \dots$. Now $V_3, V_5, \dots, V_{2k+1}, \dots$ are polynomial with rational coefficients in the coefficients of $X(x, y)$ and $Y(x, y)$. Theoretically, by using Hilbert's basis theorem, the ideal generated by these polynomials has a finite basis B_1, B_2, \dots, B_m . Hence, we have a finite set of necessary and sufficient conditions for a center, i.e., $B_i = 0$ for $i = 1, 2, \dots, M$. To calculate this basis, we reduce each V_{2k+1} modulo $\ll V_3, V_5, \dots, V_{2k-1} \gg$, the ideal generated by $V_3, V_5, \dots, V_{2k-1}$. The elements of the basis thus obtained are called the Lyapunov quantities or the focal quantities. The origin is said to be an k -order fine focus (or a focus of multiplicity k) of (E) if the first $k - 1$ Lyapunov quantities are 0 but the k -order one is not.

The above statement tell us that the solution of the center-focus for a particular system, the procedure is as follows: Compute several Lyapunov constants and when we get one significant constant that is zero, try to prove that the system obtained indeed has a center. Unfortunately, the described method has the following questions.

(1) How can we be sure that you have computed enough Lyapunov constants?

(2) How do we prove that some system candidate to have a center actually has a center?

(3) Do you know the general construction of Lyapunov constants in order to get general shortened expressions for Lyapunov constants V_3, V_5, \dots .

In Chapter 2 we devote to give possible answer for these questions.

In addition, we shall consider the following two problems.

Problem of center-focus at infinite singular point

A real planar polynomial vector field V can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space \mathcal{R}^3 with coordinates X, Y, Z . The center projection of the vector field V on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. There exists an analytic vector field $p(V)$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one constructed above from the polynomial vector field. The projection of the closed northern hemisphere H^+ of S^2 on $Z = 0$ under $(X, Y, Z) \rightarrow (X, Y)$ is called the *Poincaré disc*. A singular point q of $p(V)$ is called an *infinite* (or finite) singular point if $q \in S^1$ (or $q \in S^2/S^1$). The vector field $p(V)$ restricted to the upper hemisphere completed with the equator is called *Poincaré compactification of a polynomial vector field*.

If a real polynomial vector field has no real singular point in the equator Γ_∞ of the *Poincaré disc* and Γ_∞ can be seen as an orbit. All orbits in an inner neighborhood of Γ_∞ are spirals or closed orbits, then Γ_∞ is called the equator cycle of the vector field. Γ_∞ can become a point by using the Bendixson reciprocal radius transformation. This point is called infinity of the system. For infinity, there exists the problem of the characterization of center for concrete families of planar polynomial (or analytic) systems. In Chapter 5, we introduce corresponding research results.

Problem of center-focus at a multiple singular point

The center-focus problem for a multiple (degenerate) singular point is essentially difficult problems. There are only a few results on this direction before 2000. This book shall give some basic results in Chapter 6.

II. Small-amplitude limit cycles created by multiple Hopf bifurcations

So called *Hopf bifurcation*, it means that a differential system exhibits the phenomenon that the appearance of periodic solution (or limit cycle in plane) branching off from an equilibrium point of the system when certain changes of the parameters occur. Hopf's original work on this subject appeared in 1942, in which the author considered higher dimensional (greater than two) systems. Before 1940s, Andronov and his co-workers had done the pioneering work for planar dynamical systems. Bautin showed that for planar quadratic systems at most three small-amplitude limit cycles can bifurcate out of one equilibrium point. By the work of Andronov et al, it is well known that the bifurcation of several limit cycles from a fine focus is directly related with the stability of the focus. The sign of the first nonvanishing Lyapunov constant determines the stability of the focus. Furthermore, the number of the leading V_{2i+1}' 's ($i = 1, 2, \dots$) which vanish simultaneously is the number of limit cycles which may bifurcate from the focus. This is the reason why the investigation of the bifurcation of limit cycles deal with the computation of Lyapunov constants.

The appearance of more than one limit cycles from one equilibrium point is called *multiple Hopf bifurcation*. How these small-amplitude limit cycles can be generated?

The idea is to start with a system (E) for which the origin is a k -th weak focus, then to make a sequence of perturbations of the coefficients of $X(x, y)$ and $Y(x, y)$ each of which reverses the stability of the origin, thereby causing a limit cycle to bifurcate.

In Chapter 3 and Chapter 9 the readers shall see a lot of examples of systems having multiple Hopf bifurcation.

III. Local and non-local bifurcations of Z_q -equivariant perturbed planar Hamiltonian vector fields

The second part of Hilbert's 16th problem deals with the maximum number $H(n)$ and relative positions of limit cycles of a polynomial system

$$\frac{dx}{dt} = P_n(x, y), \quad \frac{dy}{dt} = Q_n(x, y), \quad (E_n)$$

of degree n , i.e., $\max(\deg P, \deg Q) = n$. Hilbert conjectured that the number of limit cycles of (E_n) is bounded by a number depending only on the degree n of the vector fields.

Without any doubt, the most famous one of the classical problems on planar polynomial vector fields is the second part of Hilbert's 16th problem. This is a doubly global problem: It involves the behavior of systems in the whole plane, even at infinity, and this for the whole class of systems defined by polynomials of a fixed degree n . Not only is this problem unsolved even in the case of quadratic systems, i.e. for $n = 2$, but it is still unproved that the uniform upper bound of the numbers of limit cycles occurring in quadratic systems is finite. This in spite of the fact that no one was ever able to construct an example of a quadratic system for which more than four limit cycles can be proven to exist.

Let χ_N be the space of planar vector fields $X = (P_n = \sum_{i+j=0}^n a_{ij}x^i y^j, Q_n = \sum_{i+j=0}^n b_{ij}x^i y^j)$ with the coefficients $(a_{ij}, b_{ij}) \in B \subset R^N$, for $0 \leq i + j \leq n$, $N = (n+1)(n+2)$. The standard procedure in the study of polynomial vector fields is to consider their behavior at infinity by extension to the Poincaré sphere. Thus, we can see (E_n) as an analytic N -parameter family of differential equations on S^2 with the compact base B . Then, the second part of Hilbert's 16th problem may be splitted into three parts:

Problem A Prove the finiteness of the number of limit cycles for any concrete system $X \in \chi_N$ (given a particular choice for coefficients of (E_n) i.e.,

$$\#\{L.C. \text{ of } (E_n)\} < \infty.$$

Problem B Prove for every n the existence of an uniformly bounded upper bound for the number of limit cycles on the set B as the function of the parameters, i.e.,

$$\forall n, \forall (a_{ij}, b_{ij}) \in B, \exists H(n) \text{ such that } \#\{L.C. \text{ of } (E_n)\} \leq H(n),$$

and find an upper estimate for $H(n)$.

Problem C For every n and known $K = H(n)$, find all possible configurations (or schemes) of limit cycles for every number $K, K-i, i = 1, 2, \dots, K-1$ respectively.

Hence, the second part of Hilbert's 16th problem consists of **Problems A~Problem C**.

The Problem A for polynomial and analytic differential equations are already solved by J.Ecalé [1992] and Yu.Ilyashenko [1991] independently. Of course, as S. Small stated that "These two papers have yet to be thoroughly digested by mathematical community".

Up to now, there is no approach to the solution of the Problem B, even for $n = 2$, which seem to be very complicated. But there exists a similar problem, which seems to be a little bit easier. It is the weakened Hilbert's 16th problem proposed by Arnold [1977]:

"Let H be a real polynomial of degree n and let P be a real polynomial of degree m in the variables (x, y) . How many real zeroes can the function

$$I(h) = \iint_{H \leq h} P dx dy$$

have ? "

The question is why zeroes of the Abelian integrals $I(h)$ is concerned with the second part of Hilbert's 16th problem ?

Let $H(x, y)$ be a real polynomial of degree n , and let $P(x, y)$ and $Q(x, y)$ be real polynomials of degree m . We consider a perturbed Hamiltonian system in the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} + \varepsilon P(x, y, \lambda), \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x} + \varepsilon Q(x, y, \lambda), \quad (E_H)$$

in which we assume that $0 < \varepsilon \ll 1$ and the level curves

$$H(x, y) = h$$

of the Hamiltonian system $(E_H)_{\varepsilon=0}$ contain at least a family Γ_h of closed orbits for $h \in (h_1, h_2)$.

Consider the Abelian integrals

$$I(h) = \int_{\Gamma_h} P dy - Q dx = \iint_{H \leq h} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy.$$

Poincaré-Pontrjagin-Andronov Theorem on the global center bifurcation

The following statements hold.

(i) If $I(h^*) = 0$ and $I'(h^*) \neq 0$, then there exists a hyperbolic limit cycle L_{h^*} of system (6.1) such that $L_{h^*} \rightarrow \Gamma_{h^*}$ as $\varepsilon \rightarrow 0$; and conversely, if there exists a hyperbolic limit cycle L_{h^*} of system (E_H) such that $L_{h^*} \rightarrow \Gamma_{h^*}$ as $\varepsilon \rightarrow 0$, then $I(h^*) = 0$, where $h^* \in (h_1, h_2)$.

(ii) If $I(h^*) = I'(h^*) = I''(h^*) = \dots = I^{(k-1)}(h^*) = 0$, and $I^{(k)}(h^*) \neq 0$, then (E_H) has at most k limit cycles for ε sufficiently small in the vicinity of Γ_{h^*} .

(iii) The total number of isolated zeroes of the Abelian integral (taking into account their multiplicity) is an upper bound for the number of limit cycles of system (E_H) that bifurcate from the periodic orbits of a period annulus of Hamiltonian system $(E_H)_{\varepsilon=0}$.

This theorem tells us that the weakened Hilbert's 16th problem posed by Arnold [1977] is closely related to the problem of determining an upper bound $N(n, m) = N(n, m, H, P, Q)$ for the number of limit cycles in a period annulus for the Hamiltonian system of degree $n - 1$ under the perturbations of degree m , i.e., of determining the cyclicity on a period annulus. Since the problem is concerned with the number of limit cycles that occur in systems which are close to integrable ones (only a class of subsystems of all polynomial systems). So that it is called the weakened Hilbert's 16th problem.

A closed orbit Γ_{h^*} satisfying the above theorem (i) is called a generating cycle.

To obtain Poincaré-Pontrjagin-Andronov Theorem, the problem for investigating the bifurcated limit cycles is based on the Poincaré return mapping. It is reduced to counting the number of zeroes of the displacement function

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots + \varepsilon^k M_k(h) + \cdots,$$

where $d(h, \varepsilon)$ is defined on a section to the flow, which is parameterized by the Hamiltonian value h . $I(h)$ just is equal to $M_1(h)$. The function $M_k(h)$ is called k -order Melnikov function. If $I(h) = M_1(h) \equiv 0$, we need to estimate the number of zeroes of higher order Melnikov functions. The zeroes of the first nonvanishing Melnikov function $M_k(h)$ determine the limit cycles in (E_H) emerging from periodic orbits of the Hamiltonian system $(E_H)_\varepsilon$.

In Chapter 8, we discuss a class of particular polynomial vector fields: Z_q -equivariant perturbed planar Hamiltonian vector fields, by using Poincaré-Pontrjagin-Andronov Theorem and Melnikov's result. The aim is to get some information for the studies of the second part of Hilbert's 16th problem.

IV. Isochronous center problem and periodic map

Suppose that system (E) has a center in the origin $(0, 0)$. Then, there is a family of periodic orbits of (E) enclosing the origin. The largest neighborhood of the center entirely covered by periodic orbits is called a *period annulus* of the center. If the period of the orbits is constant for all periodic orbits lying in the period annulus of the origin, then the center $(0, 0)$ is called an *isochronous center*. It has been proved that the isochronous center can exist if the period annulus of the center is unbounded.

If the origin is not an isochronous center, for a point $(\xi, 0)$ in a small neighborhood of the origin $(0, 0)$, we define $P(\xi)$ to be the minimum period of the periodic orbit passing through $(\xi, 0)$. The study for the period function $\xi \rightarrow P(\xi)$ is also very interesting problem, since monotonicity of the period function is a non-degeneracy condition for the bifurcation of subharmonic solutions of periodically forced integrable systems.

The history of the work on period functions goes back at least to 1673 when C. Huygens observed that the pendulum clock has a monotone period function and

therefore oscillates with a shorter period when the energy is decreased, i.e., as the clock spring unwinds. He hope to design a clock with isochronous oscillations in order to have a more accurate clock to be used in the navigation of ships. His solution, the cycloidal pendulum, is perhaps the first example of nonlinear isochronous center.

In the last three decades of the 20th century, a considerable number of papers of the study for isochronous centers and period maps has been published. But, for a given polynomial vector field of the degree is more than two, the characterization of isochronous center is still a very difficult, challenging and unsolved problem.

In Chapter 4, we introduce some new method to treat these problems.

Except the mentioned seven chapters, we add three chapters to introduce our more recent study results.

In Chapter 7, we consider a class of nonanalytic systems which is called “quasi-analytic systems”. We completely solve its center and isochronous center problems as well as the bifurcation of limit cycles.

In Chapter 8, as an example, for a class of Z_2 -symmetric cubic systems, we give the complete answer for the center problem and the bifurcations of limit cycles. We prove that this class of cubic systems has at least 13 limit cycles.

In the final chapter (Chapter 10), we study the center-focus problem and bifurcations of limit cycles for three-multiple nilpotent singular points. The materials are taken by our recent new papers.

We would like to cite the following words written by Anna Schlomiuk in 2004 as the finale of this preface: “Planar polynomial vector fields are dynamical systems but to perceive them uniquely from this angle is limiting, missing part of their essence and hampering development of their theory. Indeed, as dynamical systems they are very special systems and the prevalent generic viewpoint pushes them on the side. This may explain in part why Hilbert’s 16th problem as well as other problems are still unsolved even in their simple case, the quadratic one. But, Poincaré’s work shows that he regarded these systems as interesting object of study from several viewpoints, and his appreciation of the work of Darboux which he qualifies as ‘admirable’ emphasizes this point. This area is rich with problems, very hard, it is true, but exactly for this reason an open mind and a free flow of ideas is necessary. It is to be hoped that in the future there will be a better understanding of this area which lies at a crossroads of dynamical systems, algebra, geometry and where algebraic and geometric problems go hand in hand with those of dynamical systems.”

The book is intended for graduate students, post-doctors and researchers in dynamical systems. For all engineers who are interested the theory of dynamical systems, it is also a reasonable reference. It requires a minimum background of an one-year course on nonlinear differential equations.

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