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SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

of

LINEAR ALGEBRA

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BY

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Preface

Linear algebra has in recent years become an essential part of the mathematical background required of mathematicians, engineers, physicists and other scientists. This requirement reflects the importance and wide applications of the subject matter.

This book is designed for use as a textbook for a formal course in linear algebra or as a supplement to all current standard texts. It aims to present an introduction to linear algebra which will be found helpful to all readers regardless of their fields of specialization. More material has been included than can be covered in most first courses. This has been done to make the book more flexible, to provide a useful book of reference, and to stimulate further interest in the subject.

Each chapter begins with clear statements of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems are included among the solved problems. The supplementary problems serve as a complete review of the material of each chapter.

The first three chapters treat of vectors in Euclidean space, linear equations and matrices. These provide the motivation and basic computational tools for the abstract treatment of vector spaces and linear mappings which follow. A chapter on eigenvalues and eigenvectors, preceded by determinants, gives conditions for representing a linear operator by a diagonal matrix. This naturally leads to the study of various canonical forms, specifically the triangular, Jordan and rational canonical forms. In the last chapter, on inner product spaces, the spectral theorem for symmetric operators is obtained and is applied to the diagonalization of real quadratic forms. For completeness, the appendices include sections on sets and relations, algebraic structures and polynomials over a field.

I wish to thank many friends and colleagues, especially Dr. Martin Silverstein and Dr. Hwa Tsang, for invaluable suggestions and critical review of the manuscript. I also want to express my gratitude to Daniel Schaum and Nicola Monti for their very helpful cooperation.

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Chapter 1

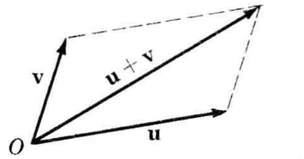
Vectors in $\mathbf{R}^n$ and $\mathbf{C}^n$

INTRODUCTION

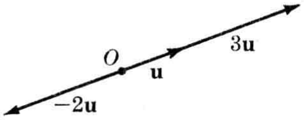
In various physical applications there appear certain quantities, such as temperature and speed, which possess only "magnitude". These can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, which possess both "magnitude" and "direction". These quantities can be represented by arrows (having appropriate lengths and directions and emanating from some given reference point O) and are called *vectors*. In this chapter we study the properties of such vectors in some detail.

We begin by considering the following operations on vectors.

(i) *Addition*: The resultant $\mathbf{u}+\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ is obtained by the so-called parallelogram law, i.e. $\mathbf{u}+\mathbf{v}$ is the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$ as shown on the right.



(ii) *Scalar multiplication*: The product $k\mathbf{u}$ of a real number k by a vector $\mathbf{u}$ is obtained by multiplying the magnitude of $\mathbf{u}$ by k and retaining the same direction if $k \geq 0$ or the opposite direction if $k < 0$, as shown on the right.



Now we assume the reader is familiar with the representation of the points in the plane by ordered pairs of real numbers. If the origin of the axes is chosen at the reference point O above, then every vector is uniquely determined by the coordinates of its endpoint. The relationship between the above operations and endpoints follows.

(i) *Addition*: If (a, b) and (c, d) are the endpoints of the vectors $\mathbf{u}$ and $\mathbf{v}$, then $(a+c, b+d)$ will be the endpoint of $\mathbf{u}+\mathbf{v}$, as shown in Fig. (a) below.

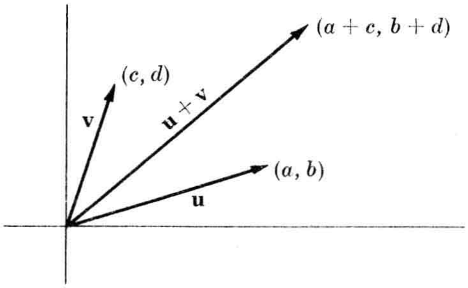


Fig. (a)

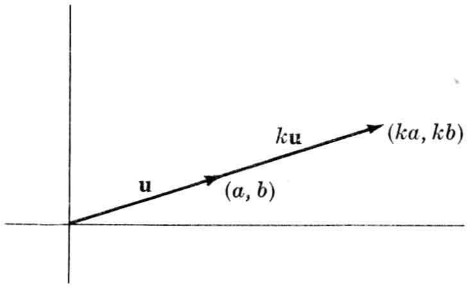


Fig. (b)

(ii) *Scalar multiplication*: If (a, b) is the endpoint of the vector $\mathbf{u}$, then (ka, kb) will be the endpoint of the vector $k\mathbf{u}$, as shown in Fig. (b) above.

Mathematically, we identify a vector with its endpoint; that is, we call the ordered pair (a, b) of real numbers a vector. In fact, we shall generalize this notion and call an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers a vector. We shall again generalize and permit the coordinates of the n-tuple to be complex numbers and not just real numbers. Furthermore, in Chapter 4, we shall abstract properties of these n-tuples and formally define the mathematical system called a *vector space.*

We assume the reader is familiar with the elementary properties of the real number field which we denote by **R**.

VECTORS IN $\mathbf{R}^n$

The set of all n-tuples of real numbers, denoted by $\mathbf{R}^n$, is called *n-space.* A particular n-tuple in $\mathbf{R}^n$, say

$$u = (u_1, u_2, \ldots, u_n)$$

is called a *point* or *vector*; the real numbers u_i are called the *components* (or: *coordinates*) of the vector u. Moreover, when discussing the space $\mathbf{R}^n$ we use the term *scalar* for the elements of **R**, i.e. for the real numbers.

Example 1.1: Consider the following vectors:

$$(0, 1), \quad (1, -3), \quad (1, 2, \sqrt{3}, 4), \quad (-5, \tfrac{1}{2}, 0, \pi)$$

The first two vectors have two components and so are points in $\mathbf{R}^2$; the last two vectors have four components and so are points in $\mathbf{R}^4$.

Two vectors u and v are *equal,* written $u = v$, if they have the same number of components, i.e. belong to the same space, and if corresponding components are equal. The vectors $(1, 2, 3)$ and $(2, 3, 1)$ are not equal, since corresponding elements are not equal.

Example 1.2: Suppose $(x - y, x + y, z - 1) = (4, 2, 3)$. Then, by definition of equality of vectors,

$$x - y = 4$$
$$x + y = 2$$
$$z - 1 = 3$$

Solving the above system of equations gives $x = 3$, $y = -1$, and $z = 4$.

VECTOR ADDITION AND SCALAR MULTIPLICATION

Let u and v be vectors in $\mathbf{R}^n$:

$$u = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad v = (v_1, v_2, \ldots, v_n)$$

The *sum* of u and v, written $u + v$, is the vector obtained by adding corresponding components:

$$u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$

The *product* of a real number k by the vector u, written ku, is the vector obtained by multiplying each component of u by k:

$$ku = (ku_1, ku_2, \ldots, ku_n)$$

Observe that $u + v$ and ku are also vectors in $\mathbf{R}^n$. We also define

$$-u = -1u \quad \text{and} \quad u - v = u + (-v)$$

The sum of vectors with different numbers of components is not defined.

Example 1.3: Let $u = (1, -3, 2, 4)$ and $v = (3, 5, -1, -2)$. Then

$$u + v = (1+3, -3+5, 2-1, 4-2) = (4, 2, 1, 2)$$
$$5u = (5 \cdot 1, 5 \cdot (-3), 5 \cdot 2, 5 \cdot 4) = (5, -15, 10, 20)$$
$$2u - 3v = (2, -6, 4, 8) + (-9, -15, 3, 6) = (-7, -21, 7, 14)$$

Example 1.4: The vector $(0, 0, \ldots, 0)$ in $\mathbf{R}^n$, denoted by 0, is called the *zero vector*. It is similar to the scalar 0 in that, for any vector $u = (u_1, u_2, \ldots, u_n)$,

$$u + 0 = (u_1 + 0, u_2 + 0, \ldots, u_n + 0) = (u_1, u_2, \ldots, u_n) = u$$

Basic properties of the vectors in $\mathbf{R}^n$ under the operations of vector addition and scalar multiplication are described in the following theorem.

Theorem 1.1: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalars $k, k' \in \mathbf{R}$:

(i) $(u+v) + w = u + (v+w)$

(ii) $u + 0 = u$

(iii) $u + (-u) = 0$

(iv) $u + v = v + u$

(v) $k(u+v) = ku + kv$

(vi) $(k+k')u = ku + k'u$

(vii) $(kk')u = k(k'u)$

(viii) $1u = u$

Remark: Suppose u and v are vectors in $\mathbf{R}^n$ for which $u = kv$ for some nonzero scalar $k \in \mathbf{R}$. Then u is said to be in the *same direction* as v if $k > 0$, and in the *opposite direction* if $k < 0$.

DOT PRODUCT

Let u and v be vectors in $\mathbf{R}^n$:

$$u = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad v = (v_1, v_2, \ldots, v_n)$$

The *dot* or *inner* product of u and v, denoted by $u \cdot v$, is the scalar obtained by multiplying corresponding components and adding the resulting products:

$$u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

The vectors u and v are said to be *orthogonal* (or: *perpendicular*) if their dot product is zero: $u \cdot v = 0$.

Example 1.5: Let $u = (1, -2, 3, -4)$, $v = (6, 7, 1, -2)$ and $w = (5, -4, 5, 7)$. Then

$$u \cdot v = 1 \cdot 6 + (-2) \cdot 7 + 3 \cdot 1 + (-4) \cdot (-2) = 6 - 14 + 3 + 8 = 3$$
$$u \cdot w = 1 \cdot 5 + (-2) \cdot (-4) + 3 \cdot 5 + (-4) \cdot 7 = 5 + 8 + 15 - 28 = 0$$

Thus u and w are orthogonal.

Basic properties of the dot product in $\mathbf{R}^n$ follow.

Theorem 1.2: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalar $k \in \mathbf{R}$:

(i) $(u+v) \cdot w = u \cdot w + v \cdot w$

(ii) $(ku) \cdot v = k(u \cdot v)$

(iii) $u \cdot v = v \cdot u$

(iv) $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$

Remark: The space $\mathbf{R}^n$ with the above operations of vector addition, scalar multiplication and dot product is usually called *Euclidean n-space*.

NORM AND DISTANCE IN $\mathbf{R}^n$

Let u and v be vectors in $\mathbf{R}^n$: $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. The *distance* between the points u and v, written $d(u, v)$, is defined by

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

The *norm* (or: *length*) of the vector u, written $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

By Theorem 1.2, $u \cdot u \geq 0$ and so the square root exists. Observe that

$$d(u, v) = \|u - v\|$$

Example 1.6: Let $u = (1, -2, 4, 1)$ and $v = (3, 1, -5, 0)$. Then

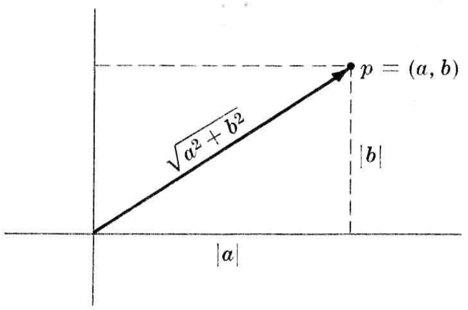
$$d(u, v) = \sqrt{(1-3)^2 + (-2-1)^2 + (4+5)^2 + (1-0)^2} = \sqrt{95}$$

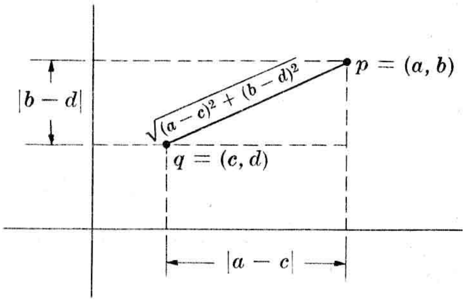
$$\|v\| = \sqrt{3^2 + 1^2 + (-5)^2 + 0^2} = \sqrt{35}$$

Now if we consider two points, say $p = (a, b)$ and $q = (c, d)$ in the plane $\mathbf{R}^2$, then

$$\|p\| = \sqrt{a^2 + b^2} \qquad \text{and} \qquad d(p, q) = \sqrt{(a-c)^2 + (b-d)^2}$$

That is, $\|p\|$ corresponds to the usual Euclidean length of the arrow from the origin to the point p, and $d(p, q)$ corresponds to the usual Euclidean distance between the points p and q, as shown below:





A similar result holds for points on the line $\mathbf{R}$ and in space $\mathbf{R}^3$.

Remark: A vector e is called a *unit vector* if its norm is 1: $\|e\| = 1$. Observe that, for any nonzero vector $u \in \mathbf{R}^n$, the vector $e_u = u/\|u\|$ is a unit vector in the same direction as u.

We now state a fundamental relationship known as the Cauchy-Schwarz inequality.

Theorem 1.3 (Cauchy-Schwarz): For any vectors $u, v \in \mathbf{R}^n$, $|u \cdot v| \leq \|u\| \, \|v\|$.

Using the above inequality, we can now define the angle θ between any two nonzero vectors $u, v \in \mathbf{R}^n$ by

$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

Note that if $u \cdot v = 0$, then $\theta = 90°$ (or: $\theta = \pi/2$). This then agrees with our previous definition of orthogonality.

COMPLEX NUMBERS

The set of complex numbers is denoted by $\mathbf{C}$. Formally, a complex number is an ordered pair (a, b) of real numbers; equality, addition and multiplication of complex numbers are defined as follows:

$$(a, b) = (c, d) \quad \text{iff} \quad a = c \text{ and } b = d$$

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

We identify the real number a with the complex number $(a, 0)$:

$$a \leftrightarrow (a, 0)$$

This is possible since the operations of addition and multiplication of real numbers are preserved under the correspondence:

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0)(b, 0) = (ab, 0)$$

Thus we view $\mathbf{R}$ as a subset of $\mathbf{C}$ and replace $(a, 0)$ by a whenever convenient and possible.

The complex number $(0, 1)$, denoted by i, has the important property that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1 \quad \text{or} \quad i = \sqrt{-1}$$

Furthermore, using the fact

$$(a, b) = (a, 0) + (0, b) \quad \text{and} \quad (0, b) = (b, 0)(0, 1)$$

we have
$$(a, b) = (a, 0) + (b, 0)(0, 1) = a + bi$$

The notation $a + bi$ is more convenient than (a, b). For example, the sum and product of complex numbers can be obtained by simply using the commutative and distributive laws and $i^2 = -1$:

$$(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

The *conjugate* of the complex number $z = (a, b) = a + bi$ is denoted and defined by

$$\bar{z} = a - bi$$

(Notice that $z\bar{z} = a^2 + b^2$.) If, in addition, $z \neq 0$, then the inverse z^{-1} of z and division by z are given by

$$z^{-1} = \frac{\bar{z}}{z\bar{z}} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i \quad \text{and} \quad \frac{w}{z} = wz^{-1}$$

where $w \in \mathbf{C}$. We also define

$$-z = -1z \quad \text{and} \quad w - z = w + (-z)$$

Example 1.7: Suppose $z = 2 + 3i$ and $w = 5 - 2i$. Then

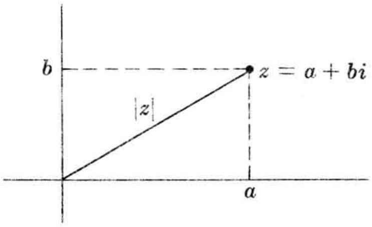
$$z + w = (2 + 3i) + (5 - 2i) = 2 + 5 + 3i - 2i = 7 + i$$

$$zw = (2 + 3i)(5 - 2i) = 10 + 15i - 4i - 6i^2 = 16 + 11i$$

$$\bar{z} = \overline{2 + 3i} = 2 - 3i \quad \text{and} \quad \bar{w} = \overline{5 - 2i} = 5 + 2i$$

$$\frac{w}{z} = \frac{5 - 2i}{2 + 3i} = \frac{(5 - 2i)(2 - 3i)}{(2 + 3i)(2 - 3i)} = \frac{4 - 19i}{13} = \frac{4}{13} - \frac{19}{13}i$$

Just as the real numbers can be represented by the points on a line, the complex numbers can be represented by the points in the plane. Specifically, we let the point (a, b) in the plane represent the complex number $z = a + bi$, i.e. whose *real part* is a and whose *imaginary part* is b. The *absolute value* of z, written $|z|$, is defined as the distance from z to the origin:



$$|z| = \sqrt{a^2 + b^2}$$

Note that $|z|$ is equal to the norm of the vector (a, b). Also, $|z| = \sqrt{z\bar{z}}$.

Example 1.8: Suppose $z = 2 + 3i$ and $w = 12 - 5i$. Then

$$|z| = \sqrt{4 + 9} = \sqrt{13} \quad \text{and} \quad |w| = \sqrt{144 + 25} = 13$$

Remark: In Appendix B we define the algebraic structure called a *field*. We emphasize that the set $\mathbf{C}$ of complex numbers with the above operations of addition and multiplication is a field.

VECTORS IN $\mathbf{C}^n$

The set of all n-tuples of complex numbers, denoted by $\mathbf{C}^n$, is called *complex n-space*. Just as in the real case, the elements of $\mathbf{C}^n$ are called *points* or *vectors*, the elements of $\mathbf{C}$ are called *scalars*, and *vector addition* in $\mathbf{C}^n$ and *scalar multiplication* on $\mathbf{C}^n$ are given by

$$(z_1, z_2, \ldots, z_n) + (w_1, w_2, \ldots, w_n) = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$$

$$z(z_1, z_2, \ldots, z_n) = (zz_1, zz_2, \ldots, zz_n)$$

where $z_i, w_i, z \in \mathbf{C}$.

Example 1.9: $(2 + 3i, 4 - i, 3) + (3 - 2i, 5i, 4 - 6i) = (5 + i, 4 + 4i, 7 - 6i)$

$2i(2 + 3i, 4 - i, 3) = (-6 + 4i, 2 + 8i, 6i)$

Now let u and v be arbitrary vectors in $\mathbf{C}^n$:

$$u = (z_1, z_2, \ldots, z_n), \qquad v = (w_1, w_2, \ldots, w_n), \qquad z_i, w_i \in \mathbf{C}$$

The *dot*, or *inner*, product of u and v is defined as follows:

$$u \cdot v = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n$$

Note that this definition reduces to the previous one in the real case, since $w_i = \bar{w}_i$ when w_i is real. The norm of u is defined by

$$\|u\| = \sqrt{u \cdot u} = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

Observe that $u \cdot u$ and so $\|u\|$ are real and positive when $u \neq 0$, and 0 when $u = 0$.

Example 1.10: Let $u = (2 + 3i, 4 - i, 2i)$ and $v = (3 - 2i, 5, 4 - 6i)$. Then

$$\begin{aligned} u \cdot v &= (2 + 3i)\overline{(3 - 2i)} + (4 - i)\overline{(5)} + (2i)\overline{(4 - 6i)} \\ &= (2 + 3i)(3 + 2i) + (4 - i)(5) + (2i)(4 + 6i) \\ &= 13i + 20 - 5i - 12 + 8i = 8 + 16i \end{aligned}$$

$$\begin{aligned} u \cdot u &= (2 + 3i)\overline{(2 + 3i)} + (4 - i)\overline{(4 - i)} + (2i)\overline{(2i)} \\ &= (2 + 3i)(2 - 3i) + (4 - i)(4 + i) + (2i)(-2i) \\ &= 13 + 17 + 4 = 34 \end{aligned}$$

$$\|u\| = \sqrt{u \cdot u} = \sqrt{34}$$

The space $\mathbf{C}^n$ with the above operations of vector addition, scalar multiplication and dot product, is called *complex Euclidean n-space*.

Remark: If $u \cdot v$ were defined by $u \cdot v = z_1w_1 + \cdots + z_nw_n$, then it is possible for $u \cdot u = 0$ even though $u \neq 0$, e.g. if $u = (1, i, 0)$. In fact, $u \cdot u$ may not even be real.

Solved Problems

VECTORS IN $\mathbf{R}^n$

1.1. Compute: (i) $(3, -4, 5) + (1, 1, -2)$; (ii) $(1, 2, -3) + (4, -5)$; (iii) $-3(4, -5, -6)$; (iv) $-(-6, 7, -8)$.

(i) Add corresponding components: $(3, -4, 5) + (1, 1, -2) = (3+1, -4+1, 5-2) = (4, -3, 3)$.

(ii) The sum is not defined since the vectors have different numbers of components.

(iii) Multiply each component by the scalar: $-3(4, -5, -6) = (-12, 15, 18)$.

(iv) Multiply each component by -1: $-(-6, 7, -8) = (6, -7, 8)$.

1.2. Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, $w = (0, 5, -8)$. Find (i) $3u - 4v$, (ii) $2u + 3v - 5w$.

First perform the scalar multiplication and then the vector addition.

(i) $3u - 4v = 3(2, -7, 1) - 4(-3, 0, 4) = (6, -21, 3) + (12, 0, -16) = (18, -21, -13)$

(ii)
$$\begin{aligned} 2u + 3v - 5w &= 2(2, -7, 1) + 3(-3, 0, 4) - 5(0, 5, -8) \\ &= (4, -14, 2) + (-9, 0, 12) + (0, -25, 40) \\ &= (4 - 9 + 0, -14 + 0 - 25, 2 + 12 + 40) = (-5, -39, 54) \end{aligned}$$

1.3. Find x and y if $(x, 3) = (2, x + y)$.

Since the two vectors are equal, the corresponding components are equal to each other:

$$x = 2, \quad 3 = x + y$$

Substitute $x = 2$ into the second equation to obtain $y = 1$. Thus $x = 2$ and $y = 1$.

1.4. Find x and y if $(4, y) = x(2, 3)$.

Multiply by the scalar x to obtain $(4, y) = x(2, 3) = (2x, 3x)$.

Set the corresponding components equal to each other: $4 = 2x$, $y = 3x$.

Solve the linear equations for x and y: $x = 2$ and $y = 6$.

1.5. Find x, y and z if $(2, -3, 4) = x(1, 1, 1) + y(1, 1, 0) + z(1, 0, 0)$.

First multiply by the scalars x, y and z and then add:

$$\begin{aligned} (2, -3, 4) &= x(1, 1, 1) + y(1, 1, 0) + z(1, 0, 0) \\ &= (x, x, x) + (y, y, 0) + (z, 0, 0) \\ &= (x + y + z, x + y, x) \end{aligned}$$

Now set the corresponding components equal to each other:

$$x + y + z = 2, \quad x + y = -3, \quad x = 4$$

To solve the system of equations, substitute $x = 4$ into the second equation to obtain $4 + y = -3$ or $y = -7$. Then substitute into the first equation to find $z = 5$. Thus $x = 4$, $y = -7$, $z = 5$.

1.6. Prove Theorem 1.1: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalars $k, k' \in \mathbf{R}$,

(i) $(u + v) + w = u + (v + w)$	(v) $k(u + v) = ku + kv$
(ii) $u + 0 = u$	(vi) $(k + k')u = ku + k'u$
(iii) $u + (-u) = 0$	(vii) $(kk')u = k(k'u)$
(iv) $u + v = v + u$	(viii) $1u = u$

Let u_i, v_i and w_i be the ith components of u, v and w, respectively.

(i) By definition, $u_i + v_i$ is the ith component of $u + v$ and so $(u_i + v_i) + w_i$ is the ith component of $(u + v) + w$. On the other hand, $v_i + w_i$ is the ith component of $v + w$ and so $u_i + (v_i + w_i)$ is the ith component of $u + (v + w)$. But u_i, v_i and w_i are real numbers for which the associative law holds, that is,

$$(u_i + v_i) + w_i = u_i + (v_i + w_i) \qquad \text{for } i = 1, \ldots, n$$

Accordingly, $(u + v) + w = u + (v + w)$ since their corresponding components are equal.

(ii) Here, $0 = (0, 0, \ldots, 0)$; hence

$$\begin{aligned} u + 0 &= (u_1, u_2, \ldots, u_n) + (0, 0, \ldots, 0) \\ &= (u_1 + 0, u_2 + 0, \ldots, u_n + 0) = (u_1, u_2, \ldots, u_n) = u \end{aligned}$$

(iii) Since $-u = -1(u_1, u_2, \ldots, u_n) = (-u_1, -u_2, \ldots, -u_n)$,

$$\begin{aligned} u + (-u) &= (u_1, u_2, \ldots, u_n) + (-u_1, -u_2, \ldots, -u_n) \\ &= (u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n) = (0, 0, \ldots, 0) = 0 \end{aligned}$$

(iv) By definition, $u_i + v_i$ is the ith component of $u + v$, and $v_i + u_i$ is the ith component of $v + u$. But u_i and v_i are real numbers for which the commutative law holds, that is,

$$u_i + v_i = v_i + u_i, \qquad i = 1, \ldots, n$$

Hence $u + v = v + u$ since their corresponding components are equal.

(v) Since $u_i + v_i$ is the ith component of $u + v$, $k(u_i + v_i)$ is the ith component of $k(u + v)$. Since ku_i and kv_i are the ith components of ku and kv respectively, $ku_i + kv_i$ is the ith component of $ku + kv$. But k, u_i and v_i are real numbers; hence

$$k(u_i + v_i) = ku_i + kv_i, \qquad i = 1, \ldots, n$$

Thus $k(u + v) = ku + kv$, as corresponding components are equal.

(vi) Observe that the first plus sign refers to the addition of the two scalars k and k' whereas the second plus sign refers to the vector addition of the two vectors ku and $k'u$.

By definition, $(k + k')u_i$ is the ith component of the vector $(k + k')u$. Since ku_i and $k'u_i$ are the ith components of ku and $k'u$ respectively, $ku_i + k'u_i$ is the ith component of $ku + k'u$. But k, k' and u_i are real numbers; hence

$$(k + k')u_i = ku_i + k'u_i, \qquad i = 1, \ldots, n$$

Thus $(k + k')u = ku + k'u$, as corresponding components are equal.

(vii) Since $k'u_i$ is the ith component of $k'u$, $k(k'u_i)$ is the ith component of $k(k'u)$. But $(kk')u_i$ is the ith component of $(kk')u$ and, since k, k' and u_i are real numbers,

$$(kk')u_i = k(k'u_i), \qquad i = 1, \ldots, n$$

Hence $(kk')u = k(k'u)$, as corresponding components are equal.

(viii) $1 \cdot u = 1(u_1, u_2, \ldots, u_n) = (1u_1, 1u_2, \ldots, 1u_n) = (u_1, u_2, \ldots, u_n) = u$.

1.7. Show that $0u = 0$ for any vector u, where clearly the first 0 is a scalar and the second 0 a vector.

Method 1: $0u = 0(u_1, u_2, \ldots, u_n) = (0u_1, 0u_2, \ldots, 0u_n) = (0, 0, \ldots, 0) = 0$

Method 2: By Theorem 1.1, $0u = (0 + 0)u = 0u + 0u$

Adding $-0u$ to both sides gives us the required result.

DOT PRODUCT

1.8. Compute $u \cdot v$ where: (i) $u = (2, -3, 6)$, $v = (8, 2, -3)$; (ii) $u = (1, -8, 0, 5)$, $v = (3, 6, 4)$; (iii) $u = (3, -5, 2, 1)$, $v = (4, 1, -2, 5)$.

(i) Multiply corresponding components and add: $u \cdot v = 2 \cdot 8 + (-3) \cdot 2 + 6 \cdot (-3) = -8$.

(ii) The dot product is not defined between vectors with different numbers of components.

(iii) Multiply corresponding components and add: $u \cdot v = 3 \cdot 4 + (-5) \cdot 1 + 2 \cdot (-2) + 1 \cdot 5 = 8$.

1.9. Determine k so that the vectors u and v are orthogonal where
(i) $u = (1, k, -3)$ and $v = (2, -5, 4)$
(ii) $u = (2, 3k, -4, 1, 5)$ and $v = (6, -1, 3, 7, 2k)$

In each case, compute $u \cdot v$, set it equal to 0, and solve for k.

(i) $u \cdot v = 1 \cdot 2 + k \cdot (-5) + (-3) \cdot 4 = 2 - 5k - 12 = 0, \quad -5k - 10 = 0, \quad k = -2$

(ii) $u \cdot v = 2 \cdot 6 + 3k \cdot (-1) + (-4) \cdot 3 + 1 \cdot 7 + 5 \cdot 2k$
$= 12 - 3k - 12 + 7 + 10k = 0, \quad k = -1$

1.10. Prove Theorem 1.2: For any vectors $u, v, w \in \mathbf{R}^n$ and any scalar $k \in \mathbf{R}$,
(i) $(u+v) \cdot w = u \cdot w + v \cdot w$ (iii) $u \cdot v = v \cdot u$
(ii) $(ku) \cdot v = k(u \cdot v)$ (iv) $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$

Let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$.

(i) Since $u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$,

$$\begin{aligned}(u+v) \cdot w &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\ &= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + \cdots + u_nw_n + v_nw_n \\ &= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n) \\ &= u \cdot w + v \cdot w\end{aligned}$$

(ii) Since $ku = (ku_1, ku_2, \ldots, ku_n)$,

$$(ku) \cdot v = ku_1v_1 + ku_2v_2 + \cdots + ku_nv_n = k(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = k(u \cdot v)$$

(iii) $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n = v_1u_1 + v_2u_2 + \cdots + v_nu_n = v \cdot u$

(iv) Since u_i^2 is nonnegative for each i, and since the sum of nonnegative real numbers is nonnegative,

$$u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$$

Furthermore, $u \cdot u = 0$ iff $u_i = 0$ for each i, that is, iff $u = 0$.

DISTANCE AND NORM IN $\mathbf{R}^n$

1.11. Find the distance $d(u, v)$ between the vectors u and v where: (i) $u = (1, 7)$, $v = (6, -5)$; (ii) $u = (3, -5, 4)$, $v = (6, 2, -1)$; (iii) $u = (5, 3, -2, -4, -1)$, $v = (2, -1, 0, -7, 2)$.

In each case use the formula $d(u, v) = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}$.

(i) $d(u, v) = \sqrt{(1-6)^2 + (7+5)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

(ii) $d(u, v) = \sqrt{(3-6)^2 + (-5-2)^2 + (4+1)^2} = \sqrt{9 + 49 + 25} = \sqrt{83}$

(iii) $d(u, v) = \sqrt{(5-2)^2 + (3+1)^2 + (-2+0)^2 + (-4+7)^2 + (-1-2)^2} = \sqrt{47}$

1.12. Find k such that $d(u, v) = 6$ where $u = (2, k, 1, -4)$ and $v = (3, -1, 6, -3)$.

$$(d(u, v))^2 = (2-3)^2 + (k+1)^2 + (1-6)^2 + (-4+3)^2 = k^2 + 2k + 28$$

Now solve $k^2 + 2k + 28 = 6^2$ to obtain $k = 2, -4$.

1.13. Find the norm $\|u\|$ of the vector u if (i) $u = (2, -7)$, (ii) $u = (3, -12, -4)$.

In each case use the formula $\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$.

(i) $\|u\| = \sqrt{2^2 + (-7)^2} = \sqrt{4 + 49} = \sqrt{53}$

(ii) $\|u\| = \sqrt{3^2 + (-12)^2 + (-4)^2} = \sqrt{9 + 144 + 16} = \sqrt{169} = 13$

1.14. Determine k such that $||u|| = \sqrt{39}$ where $u = (1, k, -2, 5)$.

$$||u||^2 = 1^2 + k^2 + (-2)^2 + 5^2 = k^2 + 30$$

Now solve $k^2 + 30 = 39$ and obtain $k = 3, -3$.

1.15. Show that $||u|| \geqq 0$, and $||u|| = 0$ iff $u = 0$.

By Theorem 1.2, $u \cdot u \geqq 0$, and $u \cdot u = 0$ iff $u = 0$. Since $||u|| = \sqrt{u \cdot u}$, the result follows.

1.16. Prove Theorem 1.3 (Cauchy-Schwarz):

For any vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbf{R}^n$, $|u \cdot v| \leqq ||u|| \, ||v||$.

We shall prove the following stronger statement: $|u \cdot v| \leqq \sum_{i=1}^{n} |u_i v_i| \leqq ||u|| \, ||v||$.

If $u = 0$ or $v = 0$, then the inequality reduces to $0 \leqq 0 \leqq 0$ and is therefore true. Hence we need only consider the case in which $u \neq 0$ and $v \neq 0$, i.e. where $||u|| \neq 0$ and $||v|| \neq 0$. Furthermore,

$$|u \cdot v| = |u_1 v_1 + \cdots + u_n v_n| \leqq |u_1 v_1| + \cdots + |u_n v_n| = \sum |u_i v_i|$$

Thus we need only prove the second inequality.

Now for any real numbers $x, y \in \mathbf{R}$, $0 \leqq (x - y)^2 = x^2 - 2xy + y^2$ or, equivalently,

$$2xy \leqq x^2 + y^2 \qquad (1)$$

Set $x = |u_i|/||u||$ and $y = |v_i|/||v||$ in *(1)* to obtain, for any i,

$$2 \frac{|u_i|}{||u||} \frac{|v_i|}{||v||} \leqq \frac{|u_i|^2}{||u||^2} + \frac{|v_i|^2}{||v||^2} \qquad (2)$$

But, by definition of the norm of a vector, $||u|| = \sum u_i^2 = \sum |u_i|^2$ and $||v|| = \sum v_i^2 = \sum |v_i|^2$. Thus summing *(2)* with respect to i and using $|u_i v_i| = |u_i| \, |v_i|$, we have

$$2 \frac{\sum |u_i v_i|}{||u|| \, ||v||} \leqq \frac{\sum |u_i|^2}{||u||^2} + \frac{\sum |v_i|^2}{||v||^2} = \frac{||u||^2}{||u||^2} + \frac{||v||^2}{||v||^2} = 2$$

that is,

$$\frac{\sum |u_i v_i|}{||u|| \, ||v||} \leqq 1$$

Multiplying both sides by $||u|| \, ||v||$, we obtain the required inequality.

1.17. Prove Minkowski's inequality:

For any vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbf{R}^n$, $||u + v|| \leqq ||u|| + ||v||$.

If $||u + v|| = 0$, the inequality clearly holds. Thus we need only consider the case $||u + v|| \neq 0$.

Now $|u_i + v_i| \leqq |u_i| + |v_i|$ for any real numbers $u_i, v_i \in \mathbf{R}$. Hence

$$\begin{aligned} ||u + v||^2 &= \sum (u_i + v_i)^2 = \sum |u_i + v_i|^2 \\ &= \sum |u_i + v_i| \, |u_i + v_i| \leqq \sum |u_i + v_i| \, (|u_i| + |v_i|) \\ &= \sum |u_i + v_i| \, |u_i| + \sum |u_i + v_i| \, |v_i| \end{aligned}$$

But by the Cauchy-Schwarz inequality (see preceding problem),

$$\sum |u_i + v_i| \, |u_i| \leqq ||u + v|| \, ||u|| \quad \text{and} \quad \sum |u_i + v_i| \, |v_i| \leqq ||u + v|| \, ||v||$$

Thus $$||u + v||^2 \leqq ||u + v|| \, ||u|| + ||u + v|| \, ||v|| = ||u + v|| \, (||u|| + ||v||)$$

Dividing by $||u + v||$, we obtain the required inequality.

1.18. Prove that the norm in $\mathbf{R}^n$ satisfies the following laws:

$[N_1]$: For any vector u, $\|u\| \geq 0$; and $\|u\| = 0$ iff $u = 0$.

$[N_2]$: For any vector u and any scalar k, $\|ku\| = |k|\,\|u\|$.

$[N_3]$: For any vectors u and v, $\|u+v\| \leq \|u\| + \|v\|$.

$[N_1]$ was proved in Problem 1.15, and $[N_3]$ in Problem 1.17. Hence we need only prove that $[N_2]$ holds.

Suppose $u = (u_1, u_2, \ldots, u_n)$ and so $ku = (ku_1, ku_2, \ldots, ku_n)$. Then

$$\|ku\|^2 = (ku_1)^2 + (ku_2)^2 + \cdots + (ku_n)^2 = k^2u_1^2 + k^2u_2^2 + \cdots + k^2u_n^2$$
$$= k^2(u_1^2 + u_2^2 + \cdots + u_n^2) = k^2\,\|u\|^2$$

The square root of both sides of the equality gives us the required result.

COMPLEX NUMBERS

1.19. Simplify: (i) $(5+3i)(2-7i)$; (ii) $(4-3i)^2$; (iii) $\dfrac{1}{3-4i}$; (iv) $\dfrac{2-7i}{5+3i}$; (v) i^3, i^4, i^{31}; (vi) $(1+2i)^3$; (vii) $\left(\dfrac{1}{2-3i}\right)^2$.

(i) $(5+3i)(2-7i) = 10 + 6i - 35i - 21i^2 = 31 - 29i$

(ii) $(4-3i)^2 = 16 - 24i + 9i^2 = 7 - 24i$

(iii) $\dfrac{1}{3-4i} = \dfrac{(3+4i)}{(3-4i)(3+4i)} = \dfrac{3+4i}{25} = \dfrac{3}{25} + \dfrac{4}{25}i$

(iv) $\dfrac{2-7i}{5+3i} = \dfrac{(2-7i)(5-3i)}{(5+3i)(5-3i)} = \dfrac{-11-41i}{34} = -\dfrac{11}{34} - \dfrac{41}{34}i$

(v) $i^3 = i^2 \cdot i = (-1)i = -i$; $i^4 = i^2 \cdot i^2 = 1$; $i^{31} = (i^4)^7 \cdot i^3 = 1^7 \cdot (-i) = -i$

(vi) $(1+2i)^3 = 1 + 6i + 12i^2 + 8i^3 = 1 + 6i - 12 - 8i = -11 - 2i$

(vii) $\left(\dfrac{1}{2-3i}\right)^2 = \dfrac{1}{-5-12i} = \dfrac{(-5+12i)}{(-5-12i)(-5+12i)} = \dfrac{-5+12i}{169} = -\dfrac{5}{169} + \dfrac{12}{169}i$

1.20. Let $z = 2 - 3i$ and $w = 4 + 5i$. Find:

(i) $z + w$ and zw; (ii) z/w; (iii) $\bar{z}$ and $\bar{w}$; (iv) $|z|$ and $|w|$.

(i) $z + w = 2 - 3i + 4 + 5i = 6 + 2i$

$zw = (2-3i)(4+5i) = 8 - 12i + 10i - 15i^2 = 23 - 2i$

(ii) $\dfrac{z}{w} = \dfrac{2-3i}{4+5i} = \dfrac{(2-3i)(4-5i)}{(4+5i)(4-5i)} = \dfrac{-7-22i}{41} = -\dfrac{7}{41} - \dfrac{22}{41}i$

(iii) Use $\overline{a+bi} = a - bi$: $\bar{z} = \overline{2-3i} = 2 + 3i$; $\bar{w} = \overline{4+5i} = 4 - 5i$.

(iv) Use $|a+bi| = \sqrt{a^2+b^2}$: $|z| = |2-3i| = \sqrt{4+9} = \sqrt{13}$; $|w| = |4+5i| = \sqrt{16+25} = \sqrt{41}$.

1.21. Prove: For any complex numbers $z, w \in \mathbf{C}$,

(i) $\overline{z+w} = \bar{z} + \bar{w}$, (ii) $\overline{zw} = \bar{z}\,\bar{w}$, (iii) $\bar{\bar{z}} = z$.

Suppose $z = a + bi$ and $w = c + di$ where $a, b, c, d \in \mathbf{R}$.

(i) $\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i}$
$= (a+c) - (b+d)i = a + c - bi - di$
$= (a - bi) + (c - di) = \bar{z} + \bar{w}$

(ii) $\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+(ad+bc)i}$
$= (ac-bd) - (ad+bc)i = (a-bi)(c-di) = \bar{z}\,\bar{w}$

(iii) $\bar{\bar{z}} = \overline{\overline{a+bi}} = \overline{a-bi} = a - (-b)i = a + bi = z$