Higher Geometry

SECOND EDITION

Xinghe Zhou
Mingsheng Yang

SCIENCE PRESS

ALPHA SCIENCE

Higher Geometry

Second Edition

Xinghe Zhou
Mingsheng Yang

SCIENCE PRESS
Beijing, China

Alpha Science International Ltd.
Oxford, U.K.

Higher Geometry
Second Edition
238 pgs. | 134 figs. | 1 tbl.

Authors
Xinghe Zhou
Mingsheng Yang

Responsible Editor
Jing Wang

Co-Published by:

Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

and

Alpha Science International Ltd.
7200 The Quorum, Oxford Business Park North
Garsington Road, Oxford OX4 2JZ, U.K.

www.alphasci.com

ISBN 978-7-03-038798-1 (Science Press, Beijing)

Printed in China

Preface to the Second Edition

Since its publication in 2003, this textbook has received multiple awards as follows. In 2004, First Place in Distinguished Textbooks of Nanjing Normal University. In 2005, one of the best textbooks for universities and colleges in Jiangsu Province. In 2006, one of the textbooks (11th Five-Year Plan) for higher education in China. In the same year, Higher Geometry received first-class honor as distinguished course of Jiangsu Province. The authors were glad to receive all these honors. Meanwhile, we also felt tremendous pressure. From the readers and other experts who gave feedbacks, we understood their love and care for the book as well as their role in developing higher geometry education in college mathematics. The authors thus felt obliged to further improve this book.

Considering that some universities might offer Higher Geometry for freshmen or sophomores, the new edition added a preparatory section 1.1. If the course is offered for sophomores or juniors, this section can be used as a reference for them. Beside correcting some mistakes in the first edition, we will make two major changes in it: First, in the new edition certain contents were written more meticulously. For example, significant revisions were made to the sections on extended plane, homogeneous coordinates, projective plane and affine plane. Some chapters were almost completely re-written. However, as a textbook, it is difficult to find a balance between comprehensibility and meticulousness. Second, the new edition added more exercises, some of which were perfected by authors during our own teaching. According to the needs of teachers and students, contents marked with "$*$" and parts printed in smaller font can be treated as selective materials. Hence, periods needed in all would not exceed those before. For the readers convenience, the new edition added an index for the four preceding chapters.

From lecture notes to the first edition of this book, and then this new edition, the authors learned that writing a good textbook is a difficult task. Although the first edition has already taken us years of work, to finish the new edition also requires a lot. The authors realized our limitations and appreciates any correction and comment from readers and colleagues. It is the authors hope that with your kind comments, the textbook can be improved again in the future.

After teaching for scores of years, the authors are still reluctant to supply students with answers to the exercises in the textbook. However, the authors have edited "Recitation for Higher Geometry: A Guide" and "Collection of Exercises for

Higher Geometry" to accompany this textbook. These materials are only available to instructors. For information regarding how to obtain these materials, please visit the web site of this course.

After the first edition of this book came out, teachers from various parts of the country and students majored in mathematics in Nanjing Normal University have all made valuable comments on it. Prof. Hua-Ping Xiong from Shangrao Normal College, Prof. Er-Cai Chen from Nanjing Normal University have discussed many times with the authors regarding revisions of this book. In which the authors have benefited a lot, and we would like to express our appreciations for these colleagues.

Experts from "Committee of Mathematics and Statistics Teaching Guidance of Higher Education in Education Ministry(China)" have been encouraging the authors throughout the revising process. Mr. Peng Lin, Mr. Hua-Qiang Hu and Ms. Jing Wang from Science Press have been working hard at this new edition of the book. The authors would like to thank all of them for their help sincerely.

Preface to the First Edition

Higher Geometry is an important course for mathematics majors in universities. Together with Mathematical Analysis, Higher Algebra, they were called the "First 3 Advanced Courses". This course aims to help the readers broaden their knowledge regarding geometric space and learn the viewpoint of transformation groups, so that the readers can further train their rational thinking capacities, enhance their appreciations of the beauty of mathematics, and improve their mathematical skills as a whole. All these will serve as a foundation for the readers further studies and researches in geometry.

The authors have been teaching higher geometry for mathematics majors in universities, including continuous education programs for over 20 years. During the process, we have accumulated a lot of experience. We have two major motivations for writing this textbook: First, there is a pressing need for reform in higher education. For years, periods have increased, and textbooks are becoming thicker and thicker. However, the connections between different courses are becoming weaker and weaker. This is a big obstacle to training students creativity. We need to reduce class hours and change teaching contents in class so as to give students more time to teach themselves and to think independently. When the class hours are cut, we need to think carefully about what to teach and how to teach them. This textbook aims to make such attempts. Second, although the authors have learned a lot from various textbooks in years of teaching, we are a bit dissatisfied with textbooks available. Thus, we had intended to make changes to material taught in class. To do this, we went over all our class notes for highlights. We have reviewed all the contents, resources, examples and exercises especially the scientific system so as to make theoretical framework of the book concise and comprehensible.

The four preceding chapters of the book are contents for class-teaching, the focus of which is on plane projective geometry. Klein's viewpoint on transformation group is also introduced. Of course, as a course in higher geometry, besides systematically introducing projective geometry and viewpoint on transformation groups, non-euclidian geometry and foundations of geometries should be also briefly introduced. As an attempt at reform, we supplemented in the textbook the fifth chapter on trail of geometries, which provides a review of the history, introductions to famous mathematicians as well as some knowledge of geometry. This chapter is intended for after- class reading. This textbook is one of the featured textbooks of the "10th

Five-Year Plan" of Nanjing Normal University. It has been tried out in Nanjing Normal University and many other universities, colleges, continuing education programs as well. Before publication, the authors have again made revisions basing on feedbacks from the trials. The electronic files that the authors used for classroom instruction is also published with the book. The authors hope that these files will be of some help to the teachers and students while using this book.

All the textbooks will limit the creativity of teachers and students to various extents. Hence, textbooks can only be guidelines that provide the teachers and students with a platform for creation. Besides, mistakes are almost unavoidable. The authors urge the readers and teachers using this book to point out mistakes. Only in this way can this book be further improved.

The authors would like to appreciate the experts from the committee for the "10th year textbook plan" of Nanjing Normal University, who offered invaluable suggestions during the review. When the framework of this book took shape, Prof. Hua-Ping Xiong from Shangrao Normal College, Prof. Gui-Yun Bian, Prof. Yi-Gui Guan from Nanjing Normal University discussed with the authors multiple times, putting forward many constructive ideas. During the process of writing, the authors received generous help from: Professors Huai-Hui Chen, Run-Sheng Yang, Jun-Hua Li, Hong-Jun Gao, Wei-Yin Ye, Rong-Bao Tu of College of Mathematics and Computer Science in Nanjing Normal University. The students of Nanjing Normal University, 1999 and 2000 grades(majoring in pure and applied mathematics) also made many good suggestions for this textbook. Now that this book is being published, the authors would like to appreciate all of these people. Mr. Peng Lin and Mr. Bo Yang of Science Press worked passionately to help the authors with the publication items. In particular, Mr. Bo Yang contributed and suggested a lot. The authors thank them wholeheartedly for it.

Contents

Chapter 1
Projective Planes

In this chapter, we first introduce the concept of transformation and discuss some of basic geometry transformations. Then, the plane of elementary geometry is extended, projective plane is defined, the homogeneous coordinates is introduced, duality principle and Desargues' Two-Triangle theorem are given. These contents are a basis of studying plane projective geometry.

1.1 Preliminaries

It is the basic idea that using the method of geometry transformations to research problems in higher geometry. As preparation, we will briefly introduce the transformation of knowledge in this section. Orthogonal transformations, similarity transformations and affine transformations are introduced. With these basic geometry transformations, for example, we can preliminarily experience way of thinking with transformations to research geometry. In this section, many conclusions will directly be written and their proofs be omitted, some of conclusions can be proved by readers on basis of their existing knowledge, while others will be self-evident after your reading the book.

1.1.1 Transformations

Definition 1.1 Let A and B be two sets. $A \times B$ denotes the set which is composed of all the ordered dual (a, b), where $a \in A, b \in B$. Then $A \times B$ is called *Cartesian product*(*product* for short) of A and B. i.e.,

$$A \times B = \{(a, b)|a \in A, b \in B\}.$$

$A \times A$ is also replaced by A^2. For example, $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

Definition 1.2 Let A and B be two sets. $R \subseteq A \times B$. Then R is called a *binary relation*(*relation* for short) from A to B. b is called the *image* of a under the binary relation R and a is called the *inverse image* of b under the relation R if $(a, b) \in R$.

Let $R^{-1} = \{(b, a)|(a, b) \in R\}$, then $R^{-1} \subseteq B \times A$ is a binary relation from B to A and R^{-1} is called the *inverse relation* of R.

Definition 1.3 Let A, B, C be sets and $R \subseteq A \times B$, $S \subseteq B \times C$. Let

$$T = \{(a, c) | \text{ there exists } b \in B \text{ s.t. } (a, b) \in R \text{ and } (b, c) \in S\}.$$

Then T is a binary relation from A to C. We call T the *product* or *composition* of R and S. Denoted by $S \circ R$, i.e., $T = S \circ R$.

For product $T = S \circ R$, we often omit "$\circ$" and directly write as $T = SR$.

Clearly, the product of relations meets associative law, but generally it does not meet commutative law.

Let $R \subseteq A^2$. Then we use R^2 to denote the product RR. Similarly, $R^3 = RR^2, \cdots, R^n = RR^{n-1}$.

Definition 1.4 Let $R \subseteq A^2$.

(1) If $(a, a) \in R$ for $\forall a \in A$, then we call relation R *reflexive.*

(2) If $(b, a) \in R$ when $(a, b) \in R$ for $\forall a, b \in A$, then we call relation R *symmetric.*

(3) If $(a, c) \in R$ when $(a, b) \in R$ and $(b, c) \in R$ for $\forall a, b, c \in A$, then we call relation R *transitive.*

If relation R satisfies the above three conditions at the same time, then we call relation R an *equivalence relation* of A.

Given an equivalence relation R of a set A. We can classify all elements of A into equivalent classes. For $a, b \in A$, a and b belong to an equivalent class if and only if $(a, b) \in R$. Obviously, the intersection of any two distinct equivalent classes of A is empty set. The union of all equivalent classes of A is exactly equal to A.

For example, let T be the set of all triangles in plane, two triangles $\triangle \sim \triangle'$ if and only if two triangles $\triangle$ and $\triangle'$ are congruent, then the relation $\sim$ is an equivalent relation of T. We can classify all triangles in plane according to the equivalent relation $\sim$, i.e., two triangles $\triangle$ and $\triangle'$ belong to the same class if $\triangle$ and $\triangle'$ are congruent.

Definition 1.5 Let R be a relation from the set A to the set B. If for any $a \in A$, there is only one ordered pair $(a, b) \in R$, the relation R is called a *correspondence* or a *mapping*, or a *function* from A to B.

Customarily, we often use lowercase English letters or Greek letters etc. to denote the correspondence. For example, use f to denote a correspondence from A to B, and write as

$$f: A \to B.$$

If $b \in B$ is the image of $a \in A$ under the correspondence f, then we can show the fact using the sign

$$f: a \mapsto b \text{ or } f(a) = b.$$

By definition 1.5, if f is a correspondence from the set A to the set B, there is only one element b of B such that $f(a) = b$ for every element a of A.

Definition 1.6 Let f be a correspondence from the set A to the set B. f is called an *injection* or a *correspondence* from A into B if for $\forall a, b \in A$, $f(a) \neq f(b)$ when $a \neq b$.

Definition 1.7 Let f be a correspondence from the set A to the set B. f is called a *surjection* or a *correspondence* from A onto B if for $\forall b \in B$, there is always $a \in A$ such that $f(a) = b$.

Definition 1.8 Let f be a correspondence from the set A to the set B. f is called a *bijection* or a *one-to-one correspondence* from A to B if f is both an injection and a surjection.

For example, given a Cartesian orthogonal coordinate system O-$e_x e_y$ in ordinary Euclidiean plane π, we can get a one-to-one correspondence $\varphi : \pi \to \mathbf{R}^2$, $P \mapsto (x, y)$, where (x, y) is the orthogonal coordinate of the point P under the Cartesian orthogonal coordinate system O-$e_x e_y$, $\mathbf{R}^2$ is the Cartesian product of $\mathbf{R}$ and itself. The correspondence φ is also called a *Cartesian coordinate mapping* of plane π. Similarly, given a one-dimensional Cartesian coordinate system on a line, we can built a bijection between all points on the line and the real number set $\mathbf{R}$.

Definition 1.9 A correspondence f from a set A to itself is called a *transformation* of A. f is called a *one-to-one transformation* of A if f is a bijection.

For any set A, $i : A \to A, a \mapsto a$ is a one-to-one transformation of A, and i is called *identity transformation* of A.

For the transformations of A, the following conclusions are obvious.

Theorem 1.1 (1) The product of two one-to-one transformations is a one-to-one transformation; Hence the product of arbitrary finite number of one-to-one transformations is a one-to-one transformation.

(2) For any one-to-one transformation f, there is only one one-to-one transformation f^{-1} such that $ff^{-1} = f^{-1}f = i$. f^{-1} is called the inverse transformation of f.

Definition 1.10 Let f be a one-to-one transformation of the set A. The element $a \in A$ is called an *invariant element* of f if $f(a) = a$. Let all elements or subsets of A have some property(quantity, respectively) P, the property P is called an *invariant property*(an *invariant quantity*, respectively) of f if the images of all elements or the image sets of the subsets under f also have the property(quantity, respectively) P. The invariant properties and invariant quantities of f are collectively referred to as the *invariance* of f.

In this course, we mainly research the invariance of geometric elements and figures under some one-to-one transformations. In the following subsections, we briefly introduce some basic geometry transformations.

1.1.2 Orthogonal Transformations

Let π denote the Euclidean plane and also denote the set of all points of Euclidean plane. A transformation of the point set π is usually called a *point transformation.*

Definition 1.11 A point transformation φ of the Euclidean plane π is called an *orthogonal transformation* if φ preserves the distance of any two points in plane π, i.e., for arbitrary $A, B \in \pi$ and $\varphi(A) = A', \varphi(B) = B'$, we have $|AB| = |A'B'|$.

Remark In this book, without any special instruction, the word "distance" always means Euclidean distance.

Thus we have the following theorem immediately.

Theorem 1.2 An orthogonal transformation of the Euclidean plane π is a bijection and the following conclusions hold.

(1) The product of arbitrary two orthogonal transformations of the Euclidean plane π is also an orthogonal transformation.

(2) The identity transformation of the Euclidean plane π is also an orthogonal transformation.

(3) The inverse transformation of any orthogonal transformation of the Euclidean plane π exist, and it is also an orthogonal transformation.

Hence, an orthogonal transformation of the Euclidean plane π is a bijection, and it preserves the distance between arbitrary two points, i.e., "distance" is the invariance of orthogonal transformations. From this, we can deduce other invariance of orthogonal transformations.

Theorem 1.3 An orthogonal transformation of the Euclidean plane π carries three collinear points into three collinear points, carries three non-collinear points into three non-collinear points, and preserves the included angle between any two lines.

Proof Let A, B, C be three points in the plane π, φ be an orthogonal transformation of the plane π, and $\varphi(A) = A'$, $\varphi(B) = B'$, $\varphi(C) = C'$.

If A, B, C are collinear, and without loss of generality, we assume that the point B is located between the points A and C, then $|AB| + |BC| = |AC|$. By definition 1.11, we have $|A'B'| + |B'C'| = |A'C'|$. Hence A', B', C' are collinear and B' is located between A' and C'.

If A, B, C are non-collinear, we have $|AB| + |BC| > |AC|$, by definition 1.11, we immediately get $|A'B'| + |B'C'| > |A'C'|$, so A', B', C' are also non-collinear.

Now we suppose that A and C are different from B and on the two sides of $\angle B$ respectively, then A' and C' are different from B' and on the two sides of $\angle B'$ respectively. By definition 1.11, $|AB| = |A'B'|$, $|BC| = |B'C'|$ and $|CA| = |C'A'|$. Hence $\Delta ABC \cong \Delta A'B'C'$. So $\angle B = \angle B'$.

We get the following two corollaries from the proving process of theorem 1.3.

Corollary 1.1 An orthogonal transformation of the plane π carries any triangle in plane π into a triangle which is congruent with the triangle. And then, an orthogonal transformation of the plane π carries any figure in plane π into a figure which is a coincident figure with the figure.

Let O-$e_x e_y$ be a Cartesian orthogonal coordinate system in the plane π. It certainly is a configuration in the plane π, so its image O'-$e'_x e'_y$ under an orthogonal transformation of π is also a Cartesian orthogonal coordinate system in plane π. Hence we have the following corollary.

Corollary 1.2 An orthogonal transformation of the plane π carries any Cartesian orthogonal coordinate system into a Cartesian orthogonal coordinate system.

Remark An orthogonal transformation of the plane π may carries a right-handed system into a left-handed system.

Theorem 1.4 Given a Cartesian orthogonal coordinate system O-$e_x e_y$ in plane π. Then a point transformation φ of π is an orthogonal transformation if and only if the algebraic expression of φ is as follows:

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}, \\ y' = a_{21}x + a_{22}y + a_{23} \end{cases} \text{or} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}. \quad (1.1)$$

Where (x, y) and (x', y') are respectively the coordinates of P and $P' = \varphi(P)$ under the coordinate system O-$e_x e_y$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is an orthogonal matrix and is called the matrix of φ.

As shown in Fig. 1.1, the coordinate of P' under the coordinate system O'-$e'_x e'_y$ is (x, y). Let the coordinates of the point O', unit vectors e'_x and e'_y under the coordinate system O-$e_x e_y$ be respectively (a_{13}, a_{23}), (a_{11}, a_{21}) and (a_{12}, a_{22}). It is easy to prove theorem 1.4 by the theory of vectors. In Fig. 1.1, the left figure shows that φ carries a right-handed system into a right-handed system, and the right figure shows that φ carries a right-handed system into a left-handed system.

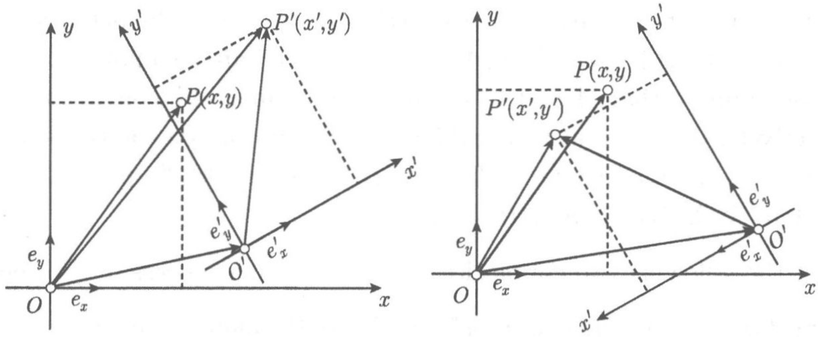

Fig. 1.1

Because A is an orthogonal matrix, $AA^{\mathrm{T}} = A^{\mathrm{T}}A = E$[①], $A^{-1} = A^{\mathrm{T}}$ and $|A| = \pm 1$. It is easy to check that: (1) φ carries a right-handed system into a right-handed system when $|A| = 1$; (2) φ carries a right-handed system into a left-handed system when $|A| = -1$.

Certainly, we must prove that the transformation (1.1) preserves the distance between any two points. Readers can prove it alone.

The transformation (1.1) is called *the first class of orthogonal transformations* or *a rigid motion* when $|A| = 1$, and the transformation (1.1) is called *the second class of orthogonal transformations* when $|A| = -1$. Obviously, the product of two same class orthogonal transformations belongs to the first class of orthogonal transformations, and the product of two distinct class orthogonal transformations belongs to the second class of orthogonal transformations.

Let the included angle between e'_x and e_x be ϑ, then the included angle between e'_y and e_x is $\vartheta + \dfrac{\pi}{2}$ or $\vartheta - \dfrac{\pi}{2}$ (see Fig. 1.1). Because e'_x and e'_y are unit vectors, (1.1) can be rewritten as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\varepsilon\sin\vartheta \\ \sin\vartheta & \varepsilon\cos\vartheta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}, \quad \text{where } \varepsilon = \pm 1. \tag{1.2}$$

Definition 1.12 A point transformation φ of plane π is called a *translation transformation*(translation for short) of π if φ moves each point for the same distance in the same direction.

Theorem 1.5 Given a Cartesian orthogonal coordinate system O-e_xe_y in plane π. Then a point transformation φ of π is a translation if and only if the algebraic expression of φ is for

$$\begin{cases} x' = x + a_{13}, \\ y' = y + a_{23} \end{cases} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}. \tag{1.3}$$

① In this book, superscript T means transposition for vector, matrix and determinant, and E denotes unit matrix.