Constantine M. Dafermos

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Hyperbolic Conservation Laws in Continuum Physics

Third Edition

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For Mihalis and Thalia

Preface to the Third Edition

The aim of this work is to present a broad overview of the theory of hyperbolic conservation laws, with emphasis on its genetic relation to classical continuum physics. It was originally published a decade ago, and a second, revised edition appeared in 2005. It is a testament to the vitality of the field that in order to keep up with recent developments it has become necessary to prepare a substantially expanded and updated new edition. A new chapter has been added, recounting the exciting recent developments in classical open problems in compressible fluid flow. Still another addition is an account of the early history of the subject, which had an interesting, tumultuous childhood. Furthermore, a substantial portion of the original text has been reorganized so as to streamline the exposition, update the information, and enrich the collection of examples. In particular, Chapter V has been completely revised. The bibliography has been updated and expanded as well, now comprising over fifteen hundred titles. The background, scope, and plan of the book are outlined in the Introduction, following this preface.

Geometric measure theory, functional analysis and dynamical systems provide the necessary tools in the theory of hyperbolic conservation laws, but to a great extent the analysis employs custom-made techniques, with strong geometric flavor, underscoring wave propagation and wave interactions. This may leave the impression that the area is insular, detached from the mainland of partial differential equations. However, the reader will soon realize that the field of hyperbolic conservation laws is far-reaching and highly diversified, as it is connected by bridges with the realms of elliptic equations, parabolic equations, equations with dispersion and the equations of the kinetic theory.

Twenty-five years ago, it might have been feasible to compose a treatise surveying the entire area; however, the explosive development of the subject over the past three decades has rendered such a goal unattainable. Thus, even though this work has encyclopedic ambitions, striving to present a panoramic view of the terrain, certain noteworthy features have been sketched very roughly or have been passed over altogether. Fortunately, a number of textbooks and specialized monographs treating some of these subjects in depth are now available. However, additional focused surveys are needed in order to compile a detailed map of the entire field.

Acknowledgments

My mentors, Jerry Ericksen and Clifford Truesdell, initiated me to continuum physics, as living scientific subject and as formal mathematical structure with fascinating history. I trust that both views are somehow reflected in this work.

I am grateful to many scientists—teachers, colleagues, and students alike—who have helped me, over the past forty years, to learn continuum physics and the theory of hyperbolic conservation laws. Since it would be impossible to list them all here by name, let me single out Stu Antman, John Ball, Alberto Bressan, Gui-Qiang Chen, Cleopatra Christoforou, Bernie Coleman, Ron DiPerna, Jim Glimm, Jim Greenberg, Mort Gurtin, Ling Hsiao, Barbara Keyfitz, Peter Lax, Philippe LeFloch, Tai-Ping Liu, Andy Majda, Piero Marcati, Ingo Müller, Walter Noll, Jim Serrin, Denis Serre, Marshall Slemrod, Joel Smoller, Luc Tartar, Konstantina Trivisa, Thanos Tzavaras, and Zhouping Xin, who have also honored me with their friendship. In particular, Denis Serre's persistent encouragement helped me to carry this arduous project to completion.

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Introduction

The seeds of continuum physics were planted with the works of the natural philosophers of the eighteenth century, most notably Euler; by the mid-nineteenth century, the trees were fully grown and ready to yield fruit. It was in this environment that the study of gas dynamics gave birth to the theory of quasilinear hyperbolic systems in divergence form, commonly called *hyperbolic conservation laws*; and these two subjects have been traveling hand in hand over the past one hundred and fifty years. This book aims at presenting the theory of hyperbolic conservation laws from the standpoint of its genetic relation to continuum physics. A sketch of the early history of this relation follows the Introduction. Even though research is still marching at a brisk pace, both fields have attained by now the degree of maturity that would warrant the writing of such an exposition.

In the realm of continuum physics, material bodies are realized as continuous media, and so-called "extensive quantities," such as mass, momentum and energy, are monitored through the fields of their densities, which are related by balance laws and constitutive equations. A self-contained, though skeletal, introduction to this branch of classical physics is presented in Chapter II. The reader may flesh it out with the help of a specialized text on the subject.

In its primal formulation, the typical balance law stipulates that the time rate of change in the amount of an extensive quantity stored inside any subdomain of the body is balanced by the rate of flux of this quantity through the boundary of the subdomain together with the rate of its production inside the subdomain. In the absence of production, a balanced extensive quantity is conserved. The special feature that renders continuum physics amenable to analytical treatment is that, under quite natural assumptions, statements of gross balance, as above, reduce to field equations, i.e., partial differential equations in divergence form.

The collection of balance laws in force demarcates and identifies particular continuum theories, such as mechanics, thermomechanics, electrodynamics, and so on. In the context of a continuum theory, constitutive equations encode the material properties of the medium, for example, heat-conducting viscous fluid, elastic solid, elastic dielectric, etc. The coupling of these constitutive relations with the field equations gives birth to closed systems of partial differential equations, dubbed "balance laws" or "conservation laws," from which the equilibrium state or motion of the continuous medium is to be determined. Historically, the vast majority of noteworthy partial differential equations were generated through that process. The central thesis of this book is that the umbilical cord joining continuum physics with the theory of partial differential equations should not be severed, as it is still carrying nourishment in both directions.

Systems of balance laws may be elliptic, typically in statics; hyperbolic, in dynamics, for media with "elastic" response; mixed elliptic-hyperbolic, in statics or dynamics, when the medium undergoes phase transitions; parabolic or mixed parabolic-hyperbolic, in the presence of viscosity, heat conductivity or other diffusive mechanisms. Accordingly, the basic notions shall be introduced, in Chapter I, at a level of generality that would encompass all of the above possibilities. Nevertheless, since the subject of this work is hyperbolic conservation laws, the discussion will eventually focus on such systems, beginning with Chapter III.

The term "homogeneous hyperbolic conservation law" refers to first-order systems of partial differential equations in divergence form,

$$(HCL) \qquad \partial_t H(U) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0,$$

that are of hyperbolic type. The state vector U, with values in $\mathbb{R}^n$, is to be determined as a function of the spatial variables $(x_1, \dots, x_m)$ and time t. The given functions H and $G_1, \dots, G_m$ are smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. The symbol ∂_t stands for $\partial/\partial t$ and ∂_α denotes $\partial/\partial x_\alpha$. The notion of hyperbolicity will be specified in Section 3.1.

Solutions to hyperbolic conservation laws may be visualized as propagating waves. When the system is nonlinear, the profiles of compression waves get progressively steeper and eventually break, generating jump discontinuities which propagate on as shocks. Hence, inevitably, the theory has to deal with weak solutions. This difficulty is compounded further by the fact that, in the context of weak solutions, uniqueness is lost. It thus becomes necessary to devise proper criteria for singling out admissible weak solutions. Continuum physics naturally induces such admissibility criteria through the Second Law of thermodynamics. These may be incorporated in the analytical theory, either directly, by stipulating outright that admissible solutions should satisfy "entropy" inequalities, or indirectly, by equipping the system with a minute amount of diffusion, which has negligible effect on smooth solutions but reacts stiffly in the presence of shocks, weeding out those that are not thermodynamically admissible. The notions of "entropy" and "vanishing diffusion," which will play a central role throughout the book, will be introduced in Chapters III and IV.

Chapter V discusses the Cauchy problem and the initial-boundary value problem for hyperbolic systems of balance laws, in the context of classical solutions. It is shown that these problems are locally well-posed and the resulting smooth solutions are stable, even within the broader class of admissible weak solutions, but their life span is finite, unless there is a dissipative source that thwarts the breaking of waves. The analysis underscores the stabilizing role of the Second Law of thermodynamics.

The Cauchy problem in the large may be considered only in the context of weak solutions. This is still terra incognita for systems of more than one equation in several space dimensions, as the analysis is at present facing seemingly insurmountable obstacles. It is even conceivable that the Cauchy problem is not generally well-posed in the realm of standard distributional weak solutions, in which case one would have to resort to the class of weaker, measure-valued solutions (see Chapter XVI). In such a setting, the hyperbolic system should be perceived as a mere shadow, in the Platonic sense, of a diffusive system with vanishing viscosity or dispersion. Nevertheless, this book will focus on success stories, namely problems admitting standard distributional weak solutions. These encompass scalar conservation laws in one or several space dimensions, systems of hyperbolic conservation laws in a single space dimension, as well as systems in several space dimensions whenever invariance (radial symmetry, stationarity, self-similarity, etc.) reduces the number of independent variables to two.

Chapter VI provides a detailed presentation of the rich and definitive theory of L^∞ and BV solutions to the Cauchy problem and the initial-boundary value problem for scalar conservation laws in several space dimensions.

Beginning with Chapter VII, the focus of the investigation is fixed on systems of conservation laws in one space dimension. In that setting, the theory has a number of special features that are of great help to the analyst, so major progress has been achieved.

Chapter VIII provides a systematic exposition of the properties of shocks. In particular, various shock admissibility criteria are introduced, compared and contrasted. Admissible shocks are then combined, in Chapter IX, with another class of particular solutions, called centered rarefaction waves, to synthesize wave fans that solve the classical Riemann problem. Solutions of the Riemann problem may in turn be employed as building blocks for constructing solutions to the Cauchy problem, in the class BV of functions of bounded variation. Two construction methods based on this approach will be presented here: the random choice scheme, in Chapter XIII, and a front tracking algorithm, in Chapter XIV. Uniqueness and stability of these solutions will also be established.

Chapter XV outlines an alternative construction of BV solutions to the Cauchy problem, for general strictly hyperbolic systems of conservation laws, by the method of vanishing viscosity.

The above construction methods generally apply when the initial data have sufficiently small total variation. This restriction seems to be generally necessary because, in certain systems, when the initial data are "large" even weak solutions to the Cauchy problem may blow up in finite time. Whether such catastrophes may occur to solutions of the field equations of continuum physics is at present a major open problem. For a limited class of systems, which however contains several important representatives, solutions with large initial data can be constructed by means of the functional analytic method of compensated compactness. This approach, which rests on the notions of measure-valued solution and the Young measure, will be outlined in Chapter XVI.

There are other interesting properties of weak solutions, beyond existence and uniqueness. In Chapter X, the notion of characteristic is extended from classical to weak solutions; and it is employed for obtaining a very precise description of regularity and long-time behavior of solutions to scalar conservation laws, in Chapter XI, as well as to systems of two conservation laws, in Chapter XII.

The final Chapter XVII deals with self-similar solutions in two space dimensions. It discusses the Riemann Problem for scalar conservation laws as well as problems of long standing in planar transonic fluid flow that have recently been solved.

In order to highlight the fundamental ideas, the discussion proceeds from the general to the particular, notwithstanding the clear pedagogical merits of the reverse course. Even so, under proper guidance, the book may also serve as a text. With that in mind, the pace of the proofs is purposely uneven: slow for the basic, elementary propositions that may provide material for an introductory course; faster for the more advanced technical results that are addressed to the experienced analyst. Even though the various parts of this work fit together to form an integral entity, readers may select a number of independent itineraries through the book. Thus, those principally interested in the conceptual foundations of the theory of hyperbolic conservation laws, in connection to continuum physics, need go through Chapters I-V only. Chapter VI, on the scalar conservation law, may be read virtually independently of the rest. Students intending to study solutions as compositions of interacting elementary waves may begin with Chapters VII-IX and then either continue on to Chapters X-XII or else pass directly to Chapter XIII and/or Chapter XIV. Similarly, Chapter XV relies solely on Chapters VII and VIII, while Chapter XVII depends on Chapters III, VII, VIII and IX. Finally, only Chapter VII is needed as a prerequisite for the functional analytic approach expounded in Chapter XVI.

Certain topics are perhaps discussed in excessive detail, as they are of special interest to the author; and a number of results are published here for the first time. On the other hand, several important aspects of the theory are barely touched upon, or are only sketched very briefly. They include the newly developed stability theory of multi-space-dimensional shocks and boundary conditions, the derivation of the balance laws of continuum physics from the kinetic theory of gases, and the study of phase transitions. Each one of these areas would warrant the writing of a specialized monograph. The most conspicuous absence is a discussion of numerics, which, beyond its practical applications, also provides valuable insight to the theory. Fortunately, a number of texts on the numerical analysis of hyperbolic conservation laws have recently appeared and may fill this gap.

A Sketch of the Early History of Hyperbolic Conservation Laws

The general theory, and even the name itself, of hyperbolic conservation laws emerged just fifty years ago, and yet the special features of this class of systems of partial differential equations had been identified long before, in the context of particular examples arising in mathematical physics. The aim here is to trace the early seminal works that launched the field and set it on its present course. A number of relevant classic papers have been collected in Johnson and Chéret [1].

The ensuing exposition will describe how the subject emerged out of fluid dynamics, how its early steps were frustrated by the confused state of thermodynamics, how it was set on a firm footing, and how it finally evolved into a special branch of the theory of partial differential equations.

This section may be read independently of the rest of the book, as it is essentially self-contained, but the student will draw extra benefit by revisiting it after getting acquainted with the current state of the art expounded in the main body of the text. Accordingly, in order to highlight the connection between past and present, the history is presented here with the benefit of hindsight: current terminology is freely used, and symbols and equations drawn from the original sources have been transliterated to modern notation.

Since the early history of hyperbolic conservation laws is inextricably intertwined with gas dynamics, we begin with a brief review of the theory of ideal gases, as it stood at the turn of the nineteenth century. Details on this topic are found in the historical tract by Truesdell [1].

The state of the ideal gas is determined by its density ρ, pressure p and (absolute) temperature θ, which are interrelated by the law associated with the names of Boyle, Gay-Lussac and Mariotte:

$$p = R\rho\theta, \tag{1}$$

where R is the universal gas constant. In the place of ρ, one may equally employ its inverse $u = 1/\rho$, namely the specific volume.

The specific heat at constant pressure or at constant volume, c_p or c_u, is the rate of change in the amount of heat stored in the gas as the temperature varies, while the pressure or the specific volume is held fixed. The ratio $\gamma = c_p/c_u$ is a constant bigger than one, called the adiabatic exponent.

Barotropic thermodynamic processes, in which $p = p(\rho)$, may be treated in the realm of mechanics, with no regard to temperature. The simplest example is an isothermal process, in which the temperature is held constant, so that, by (1),

(2) $$p = a^2\rho.$$

Subtler is the case of an isentropic or adiabatic process,[1] in which the temperature and the specific volume vary simultaneously in such a proportion that the amount of heat stored in any part of the gas remains fixed. As shown by Laplace and by Poisson, this assumption leads to

(3) $$p = a^2\rho^\gamma.$$

The oldest, and still most prominent, paradigm of a hyperbolic system of conservation laws is provided by the Euler equations for barotropic gas flow, which express the conservation of mass and momentum, relating the velocity field v with the density field and the pressure field. The pertinent publications by Euler, culminating in his definitive formulation of hydrodynamics, are collected in Euler [1], which also contains informative commentary by Truesdell. In addition to the equations that bear his name, Euler derived what is now called the Bernoulli equation for irrotational flow, so named because in steady flow it reduces to the celebrated law discovered by Daniel Bernoulli. We will encounter the aforementioned equations on several occasions in the main body of this book, beginning with Section 3.3.6.

Internal forces in an elastic fluid are transmitted by the hydrostatic pressure, which is a scalar field. As a result, the Euler equations form a system of conservation laws with distinctive geometric structure. Conservation laws of more generic type, manifesting the tensorial nature of the flux field, as discussed here in Chapter I, emerged in the 1820s from the pioneering work of Cauchy [1,2,3,4] on the theory of elasticity. Nevertheless, as we shall see below, the early work on hyperbolic conservation laws dealt almost exclusively with the one-space-dimensional setting, for which the Euler equations constitute a fully representative example.

In an important memoir on the theory of sound, published in 1808, Poisson [1] considers the Euler equations and the Bernoulli equation for rectilinear isothermal flow of an ideal gas, namely

(4) $$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2) + a^2\partial_x\rho = 0, \end{cases}$$

[1] The term "adiabatic" was coined in 1859 by Rankine, who also originated the use of the symbol γ for the adiabatic exponent. However, in the sequel we will employ the newer terminology "isentropic," while reserving "adiabatic" for a related but different use; see Section 2.5.

$$\partial_t \phi + \frac{1}{2}(\partial_x \phi)^2 + a^2 \log \rho = 0, \tag{5}$$

where ϕ is the velocity potential, $v = \partial_x \phi$. By eliminating ρ between (5) and $(4)_1$, he derives a second order equation for ϕ alone:

$$\partial_t^2 \phi + 2(\partial_x \phi)(\partial_x \partial_t \phi) + (\partial_x \phi)^2 \partial_x^2 \phi - a^2 \partial_x^2 \phi = 0. \tag{6}$$

Employing a method of solving differential equations devised by Laplace and by Legendre, he concludes that any ϕ that satisfies the functional equation

$$\partial_x \phi = f(x + (a - \partial_x \phi)t), \tag{7}$$

for some arbitrary smooth function f, is a particular solution of (6).

In current terminology, one recognizes Poisson's solution as a simple wave (see Section 7.6) on which the Riemann invariant (see Section 7.3) $v + a \log \rho$ is constant, and thus v satisfies the equation

$$\partial_t v + v \partial_x v - a \partial_x v = 0 \tag{8}$$

admitting solutions

$$v = f(x + (a - v)t), \tag{9}$$

with f an arbitrary smooth function.

Forty years after the publication of Poisson's paper, the British astronomer Challis [1] made the observation that (9), with $f(x) = -\sin(\frac{1}{2}\pi x)$, yields $v = 0$ along the straight line $x = -at$ and $v = 1$ along the straight line $x = -1 - (a - 1)t$, which raises the paradox that v must be simultaneously equal to 0 and 1 at the point $(-a, 1)$ of intersection of these straight lines. This is the earliest reference to the breakdown of classical solutions, which pervades the entire theory of hyperbolic conservation laws.

The issue raised by Challis was addressed almost immediately by Stokes [1], his colleague at the University of Cambridge. Stokes notes that, according to Poisson's solution (9), along each straight line $x = \bar{x} - (a - f(\bar{x}))t$, we have $v(x,t) = f(\bar{x})$ and

$$\partial_x v(x,t) = \frac{f'(\bar{x})}{1 + t f'(\bar{x})}. \tag{10}$$

Thus, unless f is nondecreasing, the wave will break at $t = -1/f'(\bar{x})$, where $f'(\bar{x})$ is the minimum of f'. He then ponders what would happen after singularities develop and comes up with an original and bold conjecture. In his own words: "Perhaps the most natural supposition to make for trial is that a surface of discontinuity is formed, in passing across which there is an abrupt change of density and velocity." He seems highly conscious that this is a far-reaching idea, going well beyond the particular setting of Poisson's solution, as he writes: "Although I was led to the subject by considering the interpretation of the integral (9), the consideration of a discontinuous motion is not here introduced in connection with that interpretation, but simply for its own sake; and I wish the two subjects to be considered as quite distinct."

Stokes then proceeds to characterize the jump discontinuities that conform to the governing physical laws of conservation of mass and momentum, which underlie the Euler equations in the realm of smooth flows. Assuming that density and velocity jump from (ρ_-, v_-) to (ρ_+, v_+) across a line of discontinuity propagating with speed σ (i.e., having slope σ), he shows that

$$\text{(11)} \qquad \begin{cases} \rho_+ v_+ - \rho_- v_- = \sigma(\rho_+ - \rho_-) \\ \rho_+ v_+^2 + a^2\rho_+ - \rho_- v_-^2 - a^2\rho_- = \sigma(\rho_+ v_+ - \rho_- v_-). \end{cases}$$

By eliminating σ between the above two equations, he gets

$$\text{(12)} \qquad \rho_-\rho_+(v_+ - v_-)^2 = a^2(\rho_+ - \rho_-)^2.$$

Thus, Stokes [1] introduces, in the context of the Euler equations (4) for isothermal flow, the notion of a shock wave and derives what are now known as the Rankine-Hugoniot jump conditions (see Section 8.1), which characterize distributional weak solutions of (4). This paper is one of the cornerstones of the theory of hyperbolic conservation laws. However, the development of the subject was soon to hit a roadblock.

Stokes's idea of contemplating flows with jump discontinuities was criticized, apparently in private, by Sir William Thomson (Lord Kelvin), and later by Lord Rayleigh, in private correspondence (Rayleigh [1]) as well as in print (Rayleigh [2, §253]), on the following grounds: they argued that jump discontinuities should not produce or consume mechanical energy. A calculation shows that this would require

$$\text{(13)} \qquad 2\rho_-\rho_+ \log\left(\frac{\rho_-}{\rho_+}\right) = \rho_-^2 - \rho_+^2,$$

which is incompatible with $\rho_- \neq \rho_+$.

In order to place the above argument in the present context of the theory of conservation laws, one should notice that any smooth solution of the Euler equations (4) automatically satisfies the conservation law of mechanical energy

$$\text{(14)} \qquad \partial_t\left(\tfrac{1}{2}\rho v^2 + a^2\rho\log\rho\right) + \partial_x\left(\tfrac{1}{2}\rho v^3 + a^2\rho v\log\rho + a^2\rho v\right) = 0.$$

In current terminology, $\frac{1}{2}\rho v^2 + a^2\rho\log\rho$ is an entropy for the system (4), with entropy flux $\frac{1}{2}\rho v^3 + a^2\rho v\log\rho + a^2\rho v$; see Section 7.4. Assuming that mechanical energy should be conserved, even in the presence of shocks, induces the jump condition

$$\text{(15)} \qquad \tfrac{1}{2}\rho_+ v_+^3 + a^2\rho_+ v_+\log\rho_+ + a^2\rho_+ v_+ - \tfrac{1}{2}\rho_- v_-^3 - a^2\rho_- v_-\log\rho_- - a^2\rho_- v_-$$
$$= \sigma\left[\tfrac{1}{2}\rho_+ v_+^2 + a^2\rho_+\log\rho_+ - \tfrac{1}{2}\rho_- v_-^2 - a^2\rho_-\log\rho_-\right].$$

Eliminating σ between $(11)_1$ and (15), and making use of (12), one arrives at (13).