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a cura di M. N. ROSENBLUTH
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VILLA MONASTERO
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edited by M. N. ROSENBLUTH
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Introduction.

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Unhappily, I am finally writing this introduction in the days following the awful events of President Kennedy's assassination which is dominating my thoughts now. Perhaps I will be excused then for saying some words about a strong nontechnical recollection I have of the Varenna summer school. I think everyone who has been there will recall it as a vital experience in international co-operation where new friends from many countries are made and ideas and feelings—technical, political and personal—freely exchanged. Even after making allowances for the idyllic setting and the common technical interest of the participants, one is bound to receive a strong impression of the common humanity of us all and a hope that the obstacles to the late President's vision of a world of peace and progress may indeed be overcome.

The field of plasma physics is especially suitable for an international school since so many of the participants have been involved in the world-wide quest for thermonuclear power. Fortunately, a happy tradition of total co-operation between all countries exists in this area, perhaps because it is almost unique in being a well-defined and challenging scientific problem from which great economic, but not military, benefits may be foreseen as the result of scientific progress.

My own interest in the thermonuclear problem has undoubtedly led to a certain bias in the selection of the topics to be covered in this course. It has always seemed to me that the best starting point for the study of plasma physics is to try to understand the simplest possible, most idealized, situation. This implies a high-temperature plasma in which collective effects predominate and a quiescent plasma in a simple equilibrium so that linear wave dynamics can be considered. The experimental attainment of this theoreticians' dream is of course precisely the thermonuclear problem. After this understanding has been reached it may be possible to comprehend more fully complex astrophysical and geophysical plasmas In the meantime, considerable technological progress may be made through intuition and invention, but I believe that the ultimate exploitation of plasmas will depend on systematic scientific theory and experiment.

The aim of this series of lectures then has been to present the elements of the theory of high-temperature plasmas. I believe that most of the important topics have been at least sketchily covered, with the exception of the study of the various complex types of high frequency plasma waves—plasma oscillation, cyclotron waves, whistlers, etc. All of the students had considerable knowledge of the field already so that the lectures are on a fairly advanced level. Some preliminary study of the field would probably be advisable before attempting to read these notes, although I do not feel that they should be useful only to experts. For example, the lecture notes from the Riso course in 1960 (Riso Report No. 18 Danish AEK) are somewhat more elementary. It must be pointed out that these lecture notes occupy a no-man's land between original research papers and a coherently organized and polished book of a single author. Their virtue perhaps is that they are broader and more up to date than a monograph could be.

The first step in understanding a plasma is the reduction of this many-body system to a 6-dimensional problem described by the Vlasov and Fokker-Planck equations. This topic is covered in the Kinetic Theory lectures of the noted Shakespearian actor, Dr. W. BE. THOMPSON. A further reduction to the modified magnetohydrodynamic fluid-type equations is possible in the limit of small gyro-radius and low frequency. This development culminates in the energy-principle for determining hydrodynamic stability and is discussed by one of its originators, Dr. RUSSEL KULSRUD. The energy principle has been exploited to great effect in the study of complex plasma confinement geometries. Some of these very general results are presented here by Dr. CLAUDE MERCIER. These sets of lectures may be considered to be on basic and already well-understood aspects of plasma physics.

The other lectures concern topics still imperfectly understood and under development.

An important area in which much work continues to be done is the exploitation of the rich variety of phenomena contained within the Vlasov equations but not the simple fluid equations. A superficial treatment of some of these phenomena is contained in my lectures on Microinstabilities. It should be noted that a slight extension of these techniques to include variations along the magnetic field leads to the «universal» instability.

It has become apparent that much of the content of the usual MHD equations is contained in the infinite conductivity constraint that particles remain «tied» to field lines in any motion. Thus when this constraint is relaxed even slightly new types of motion become possible as is discussed in the lectures on Resistive Instabilities by Dr. HAROLD FURTH, who also enlivened the proceedings notably in his role of Court Jester.

As in other types of fluid dynamics a great many of the answers we seek are obscured by our inability to treat nonlinear problems effectively. Some important beginnings have been made and are discussed in the lectures of Professor PETER STURROCK. Another topic, of both mathematical and physical importance, concerns the proper ordering scheme for the approximate treatment of boundary layers and is discussed in the lectures of Dr. BRUNO BERTOTTI. Finally, while the course was in general directed toward the study of high-temperature plasma a very fascinating sidelight was given to us in the lively lectures of Dr. GUNTER ECKER on the low-temperature gas-discharge regime. It is interesting to note the differences and similarities in two such closely-related topics and sobering to realize how rare and difficult communication between them has been.

In addition to these formal lectures the theory of adiabatic invariants was discussed by R. KULSRUD following two important papers by KULSRUD himself and M. KRUSKAL. They are reprinted « in extenso ». Further seminars on specialized and current research were presented by G. SANDRI on The New Foundations of Statistical Dynamics, D. PFIRSCH on Microinstabilities of the Mirror Type in Inhomogeneous Plasmas, G. LAVAL and R. PELLAT on the Boundary Layer between a Plasma and a Magnetic Field, G. KNORR on Nonlinear phenomena in Microscopic Wave Propagation, S. CUPERMAN, F. ENGELMAN and J. OXENIUS on Nonthermal Impurity Radiation from a Hot Spherical Plasma, K. VON HAGENOW and H. KOPPE on The Partition Function of a Completely Ionized Gas, F. ENGELMAN on Quantum-Mechanical Treatment of Electric Microfield Problems in Plasmas, J. B. TAYLOR on Finite Larmor Radius Effect and the Rotational Instability of Plasma in Fast B_z Compression Experiments (Theta-Pinch), D. VOSLAMBER and D. K. CALLEBAUT on Stability of Force-Free Magnetic Fields, C. F. WANDEL and O. KOFOED-HANSEN on the Eulerian-Lagrangian Transformation in the Statistical Theory of Turbulence, N. A. KRALL on Oscillations in Nonuniform Plasmas, P. J. KELLOG on The Earth's Environment: Observations on a Very Large Mirror Machine, E. T. KARLSON on the Motion of a Plasma in an Inhomogeneous Magnetic Field.

Unfortunately it is only possible to list these by title and author here as we dispose only of their summaries. These seminars helped a great deal to give some experimental back-up for these theoretical lectures.

Finally, I would like to give my deep thanks to all those who have participated so whole-heartedly in the course—to the knowledgeable attentive and long-suffering student body, to those who presented seminars; and to the lecturers. I hope the reader will concur with my judgement of the excellence of their efforts.

The chief thanks of course go to our hosts, the Italian Physical Society, who have provided the magnificent villa on Lake Como, with its excellent facilities and amenities, making a physical background most conducive to

relaxed and harmonious discussion. I would like to mention especially those representatives on the scene who have been so helpful and made us feel so welcome; Dr. BRUNO BERTOTTI, the scientific secretary, who has done most of the director's work, the secretarial staff—Misses LAFARGE, NAVONE, and VARIOLA—who not only coped with these mountains of manuscript but also went far out of their way to take care of all the troublesome details which invariably arise, and Professor G. GERMANÀ who ran the whole operation so skillfully.

INDICE

LEZIONI

SEMINARI SULL'INVARIANZA ADIABATICA

Kinetic Theory of Plasma.

W. B. THOMPSON
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PART I

1. – Introduction.

The object of kinetic theory is to extract from an exact and detailed description of a complex physical system that information needed to describe its gross behaviour.

For example, consider a system composed of many charged particles having mass m_i, charge e_i and velocity $\boldsymbol{v}_i$; then in a volume V large enough to contain many particles, we may define a macroscopic density ϱ, velocity $\boldsymbol{V}$, charge Q, and current $\boldsymbol{J}$, by

$$\frac{1}{V}\sum m_i = \varrho\,; \qquad \frac{1}{V}\sum_i m_i \boldsymbol{v}_i = \varrho \boldsymbol{V}\,;$$

$$\frac{1}{V}\sum e_i = Q\,; \qquad \frac{1}{V}\sum_i e_i \boldsymbol{v}_i = \boldsymbol{J}\,.$$

It is often convenient to write

$$\boldsymbol{v}_i = \boldsymbol{V} + \boldsymbol{c}_i\,; \qquad \boldsymbol{J} = Q\boldsymbol{V} + \boldsymbol{j}\,.$$

The macroscopic variables ϱ, $\boldsymbol{V}$ etc. satisfy equations of motion that may be deduced from the laws of motion for the individual particle, *i.e.* from

$$m\dot{\boldsymbol{v}}_i = \boldsymbol{F}_i = e_i(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}) + \boldsymbol{F}_{\text{int}}\,,$$

where

$$\sum_i \boldsymbol{F}_{\text{int}} = 0$$

from conservation of momentum. Then $\partial\varrho/\partial t$ is determined by equating the rate of change of density to the flux into a volume; *i.e.*

$$\frac{\partial\varrho}{\partial t}+\nabla\cdot(\varrho\boldsymbol{V})=0\,. \tag{I.1.1}$$

The rate of change of momentum is similarly

$$\frac{\partial}{\partial t}(\varrho\boldsymbol{V})+\frac{\partial}{\partial\boldsymbol{x}}\cdot\sum_i m\boldsymbol{v}_i\boldsymbol{v}_i=\frac{\partial}{\partial t}(\varrho\boldsymbol{V})+\frac{\partial}{\partial\boldsymbol{x}}\cdot(\varrho\boldsymbol{V}\boldsymbol{V})+\frac{\partial}{\partial\boldsymbol{x}}\cdot\sum m\,\boldsymbol{c}_i\,\boldsymbol{c}_i=$$
$$=\sum_i\boldsymbol{F}_i=\sum_i e_i(\boldsymbol{E}+\boldsymbol{v}_i\times\boldsymbol{B})=Q(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B})+\boldsymbol{j}\times\boldsymbol{B}\,,$$

and, defining the stress tensor $p_{rs}=\sum_i mc_{ir}c_{is}$ and using the continuity eq. (I.1.1),

$$\varrho\left(\frac{\partial\boldsymbol{V}}{\partial t}+\boldsymbol{V}\cdot\nabla\boldsymbol{V}\right)+\nabla\cdot\boldsymbol{p}=Q(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B})+\boldsymbol{j}\times\boldsymbol{B}\,. \tag{I.1.2}$$

In a similar way, we may equate the rate of gain of energy to the rate of doing work, *i.e.*

$$\frac{\partial}{\partial t}\left(\sum_i\frac{1}{2}\,mv_i^2\right)+\nabla\cdot\sum_i\boldsymbol{v}_i\,\frac{1}{2}\,mv_i^2=\sum_i\boldsymbol{F}_i\cdot\boldsymbol{v}_i=\sum_i e\boldsymbol{v}_i\cdot\boldsymbol{E}=\boldsymbol{J}\cdot\boldsymbol{E}\,,$$

or

$$\boldsymbol{V}\cdot\left(\varrho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}+\nabla\cdot\boldsymbol{p}\right)+\frac{\mathrm{D}}{\mathrm{D}t}\sum_i\frac{1}{2}\,mc_i^2+\sum_i\frac{1}{2}\,mc_i^2\nabla\cdot\boldsymbol{V}+(\nabla\cdot\boldsymbol{p})\cdot\boldsymbol{v}+\nabla\cdot\sum\boldsymbol{c}\,\frac{1}{2}\,mc^2=$$
$$=Q\boldsymbol{V}\cdot\boldsymbol{E}+\boldsymbol{j}\cdot\boldsymbol{E}\,.$$

and defining the internal, energy

$$U=\sum_i\tfrac{1}{2}\,mc_i^2=\tfrac{3}{2}\,nkT\,,$$

and the heat flux

$$\boldsymbol{q}=\sum\boldsymbol{c}\,\tfrac{1}{2}\,mc^2\,,$$

$$\frac{\mathrm{D}U}{\mathrm{D}t}+U\nabla\cdot\boldsymbol{V}+(\boldsymbol{p}\cdot\nabla\boldsymbol{V})+\nabla\cdot\boldsymbol{q}=\boldsymbol{j}\cdot\boldsymbol{E}\,. \tag{I.1.3}$$

For an ionized gas, the charged particles are of two sorts; electrons and ions, and the interdiffusion of these particles gives rise to a current; the electron and ions moving with speeds $\boldsymbol{V}+\Delta\boldsymbol{V}_-$, $\boldsymbol{V}+\Delta\boldsymbol{V}_+$. By considering the two gases separately the quantity $\Delta\boldsymbol{V}_-$ is found to satisfy

$$\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}+\frac{\mathrm{D}\Delta\boldsymbol{V}}{\mathrm{D}t}+\Delta\boldsymbol{V}\cdot\nabla\Delta\boldsymbol{V}+\Delta\boldsymbol{V}\cdot\boldsymbol{V}+\frac{1}{nm}\nabla\cdot\boldsymbol{p}_-=\frac{e_-}{m_-}(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}+\Delta\boldsymbol{V}\times\boldsymbol{B})+\frac{\boldsymbol{F}_-}{m_-}\,.$$

If this and a similar equation for ΔV_+ are multiplied by a product of density and charge there results

$$\frac{\mathrm{D}\boldsymbol{j}}{\mathrm{D}t}+\frac{e_-}{m_-}\nabla\cdot\boldsymbol{p}_-+\frac{e_+}{M_+}\nabla\cdot\boldsymbol{p}_+-\frac{Q}{\varrho}\nabla\cdot\boldsymbol{p}_++\left[\frac{Q^2}{\varrho}-\left[n_-\left(\frac{e^2}{m_-}\right)+n_+\left(\frac{e}{m_+}\right)^2\right]\cdot\right.$$

$$\left.\cdot[\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}]+\frac{Q}{\varrho}(\boldsymbol{j}\times\boldsymbol{B})-\left[n_-\left(\frac{e_-^2}{m_-}\right)\Delta\boldsymbol{V}_-+n_+\left(\frac{e}{M_+}\right)^2\Delta\boldsymbol{V}_+\right]\right]\times\boldsymbol{B}=0\,,$$

(I.1.4) $$\frac{m_-}{ne^2}\frac{\mathrm{D}\boldsymbol{j}}{\mathrm{D}t}+\frac{m_-}{ne_-}[\nabla\cdot\boldsymbol{p}_--\boldsymbol{j}\times\boldsymbol{B}]+[\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}]-\eta\boldsymbol{j}=0\,,$$

where

$$\eta\boldsymbol{j}=\frac{m_-}{ne^2}\left[\frac{ne_-}{m_-}\boldsymbol{F}_-+\frac{ne_+}{m_+}\boldsymbol{F}_+\right].$$

A more formal treatment of the relation between the microscopic and the macroscopic can be effected by employing a distribution function f; a quantity which describes the statistical evolution of the system, in which case the underlying microscopic dynamics of the system is embraced in an equation of motion for f, the equation of transport.

There are several sorts of distribution function f, ranging from the Liouville function $F(x_1, x_2, \ldots, x_N, v_1, \ldots, v_N)$ to the Boltzmann single-particle function $f(x, v)$. The Liouville function is a function of the complete set of micro coordinates, and satisfies the equation

(I.1.5) $$\frac{\partial F}{\partial t}+[H(x_1\ldots x_N,\ v_1\ldots v_N), F]=0\,,$$

which is completely equivalent to the microscopic dynamics, H being the complete Hamiltonian. This can be written, introducing the acceleration field $\boldsymbol{A}_i(x_1\ldots x_N)$,

$$\frac{\partial F}{\partial t}+\sum_i\left(\boldsymbol{v}_i\cdot\frac{\partial F}{\partial\boldsymbol{x}_i}+\boldsymbol{A}_i(x_1\ldots x_N)\cdot\frac{\partial F}{\partial\boldsymbol{v}}\right)=0\,.$$

The equivalence of Liouville's equation and the equations of motion is established by observing that if the system is given as in the state specified by

$$\boldsymbol{x}_1(0),\ \boldsymbol{x}_2(0)\ldots\boldsymbol{x}_N(0),\ \boldsymbol{v}_1(0),\ \boldsymbol{v}_2(0)\ldots\boldsymbol{v}_N(0)\,,$$

so that

$$F(0)=\Pi\,\delta(\boldsymbol{x}_i-\boldsymbol{x}_i(0))\,\delta(\boldsymbol{v}_i-\boldsymbol{v}_i(0))\,,$$

its subsequent evolution is described by

$$F(t) = \Pi\, \delta(\boldsymbol{x}_i - \boldsymbol{X}_i(t))\, \delta(\boldsymbol{v}_i - \boldsymbol{V}_i(t)) \,, \tag{I.1.6}$$

where $\boldsymbol{X}_i(t)$ and $\boldsymbol{V}_i(t)$ are the relevant solutions of the equations of motion, *i.e.*

$$\boldsymbol{X}_i(t) = \boldsymbol{x}_i(0) + \int_0^t \mathrm{d}t'\, \boldsymbol{V}_i(t') \,,$$

$$\boldsymbol{V}_i(t) = \boldsymbol{v}_i(0) + \int_0^t \mathrm{d}t'\, \boldsymbol{A}_i(\boldsymbol{X}_i(t'), \boldsymbol{V}_i(t');\ \sum \boldsymbol{X}_{\text{not}\,i}(t'), \boldsymbol{V}_{\text{not}\,i}(t')) \,. \tag{I.1.7}$$

The Boltzmann function on the other hand satisfies an equation of the form

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{A}_0(x, t)\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \left.\frac{\partial f}{\partial t}\right|_{\text{int}} . \tag{I.1.8}$$

The transport equation for the Boltzmann function may be obtained by repeated integration of the Liouville equation, for f itself is defined by

$$f(x_1, v_1) = V \int F(x_1 \ldots x_N, v_1 \ldots v_N)\, \mathrm{d}^3x_2 \ldots \mathrm{d}^3x_N\, \mathrm{d}^3v_2 \ldots \mathrm{d}^3v_N \,.$$

If there were no interaction between particles so that the acceleration field $\boldsymbol{A}_i$ could be factored as $\boldsymbol{A}_1(x_1, v_1)\boldsymbol{A}_2(x_2, v_2)$ ... etc., then a closed equation for f could be obtained by integrating over the Liouville equation as

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{A}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0 \,, \tag{I.1.9}$$

the collisionless Boltzmann, or Vlasov equation.

If, however, there exists an inter-particle potential $\phi(x_i, x_j)$ the final integral cannot be evaluated in terms of $\boldsymbol{x}_1$, $\boldsymbol{v}_1$ and f alone, instead it becomes

$$-\frac{V}{m_1}\sum_j \frac{\partial}{\partial \boldsymbol{v}_1}\cdot\int \frac{\partial}{\partial \boldsymbol{x}_1}\phi(\boldsymbol{x}_1, \boldsymbol{x}_j) F(\boldsymbol{x}_1 \ldots \boldsymbol{x}_N)\, \mathrm{d}^3v_2 \ldots \mathrm{d}^3v_N\, \mathrm{d}^3x_2 \ldots \mathrm{d}^3x_N \,,$$

or, introducing the two-particle function

$$f(1, 2) = V^2 \int F(\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots \boldsymbol{x}_N)\, \mathrm{d}^3x_3 \ldots \mathrm{d}^3x_N\, \mathrm{d}^3v_3 \ldots \mathrm{d}^3v_N \,,$$

$$\left.\frac{\partial f}{\partial t}\right|_{\text{int}} = \frac{N}{V m_1}\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\int \frac{\partial \phi(1, 2)}{\partial \boldsymbol{x}_1} f(1, 2)\, \mathrm{d}^3x_2\, \mathrm{d}^3v_2 \,,$$

and the general equation of transport for f becomes

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{A}_0\cdot\frac{\partial f}{\partial \boldsymbol{v}} - \frac{n}{m_1}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int\frac{\partial}{\partial \boldsymbol{x}_1}\phi(x_1, x_2)f(1, 2)\,\mathrm{d}^3x_2\,\mathrm{d}^3v_2\,,$$

or

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{A}_0\cdot\frac{\partial f}{\partial \boldsymbol{v}} + I(f) = 0\,. \qquad \text{(I.1.10)}$$

The first major problem of kinetic theory is to find an approximate form for $I(f)$; the second being that of solving (I.1.10) for a given form of I and deducing the moments required for a macroscopic description of phenomena.

For diffuse gases in which a *strong* but *localized interaction* occurs between the particles, a coarse-grained equation for f may be obtained in which the interaction term I is represented by the rate of change of f produced by impulsive collisions between particles; $(\boldsymbol{g}=\boldsymbol{v}_1-\boldsymbol{v}_2)$, $\theta=$ scattering angle

$$\left.\frac{\partial f}{\partial t}\right|_{\text{int}} = \int\mathrm{d}\Omega\int\mathrm{d}^3v_2\,g\,\sigma(g, \theta)\,[f(\bar{\boldsymbol{v}}_1)f(\bar{\boldsymbol{v}}_2) - f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)]\,,$$

where $\bar{v}_1$, $\bar{v}_2$ are related to v_1, v_2 and θ, being in fact the negatives of the velocities resulting when a collision between v_1 and v_2 occurs with a scattering angle θ. Our later work will be devoted to showing that, with certain corrections, this result is a valid approximation for an ionized gas where the forces of interaction are *weak*, but *long ranged.* At present we will concentrate on the second problem, that of solving Boltzmann's equation and determining the transport coefficients.

2. – Hydrodynamic equations from the transport equation.

As a preliminary to any attempt to solve the Boltzmann equation we will use it to form the hydrodynamic equations. To do this we use the definitions of Section 1, which, expressed in terms of the distribution function f become the following moments of f

$$\varrho = \int fm\,\mathrm{d}^3v\,; \qquad \varrho\boldsymbol{V} = \int fm\boldsymbol{v}\,\mathrm{d}^3v\,; \qquad \boldsymbol{c} = \boldsymbol{v} - \boldsymbol{V}\,;$$

$$\boldsymbol{p} = \int fm\boldsymbol{cc}\,\mathrm{d}^3v\,; \qquad \frac{3}{2}kT = \int f\,\tfrac{1}{2}mc^2\,\mathrm{d}^3v\,; \qquad \boldsymbol{q} = \int f\,\tfrac{1}{2}mc^2\,\boldsymbol{c}\,\mathrm{d}^3v\,.$$

Since the B.E. forms a representation of the dynamics of the system, the macroscopic equations for the moments may be formed therefrom, *i.e.* from

(I.2.1) $$\int m\left\{\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}}+\boldsymbol{A}\cdot\frac{\partial f}{\partial \boldsymbol{v}}-I(f)\right\}\mathrm{d}^3v=0\,,$$

$$\frac{\partial\varrho}{\partial t}+\operatorname{div}(\varrho\boldsymbol{V})=0\,,$$

since $\int mI(f)\,\mathrm{d}^3v=0$, from mass conservation.

From $\int \mathrm{d}^3 m\, m\boldsymbol{v}\{\ \}=0$,

(I.2.2) $$\varrho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}+\nabla\cdot\boldsymbol{p}-\boldsymbol{F}=0 \qquad \text{where} \qquad \boldsymbol{F}=\int m\boldsymbol{A}f\,\mathrm{d}^3v\,,$$

and (I.2.1) has been used. Finally from $\int \mathrm{d}^3v\,\frac{1}{2}\,mv^2\{\ \}$,

(I.2.3) $$\frac{\mathrm{D}U}{\mathrm{D}t}+U\operatorname{div}\boldsymbol{V}+\boldsymbol{p}:\nabla\boldsymbol{V}+\operatorname{div}\boldsymbol{q}=0\,,$$

and

$$U=\tfrac{3}{2}\,nkT\,.$$

For an ionized gas, there are similar equations for each component, although now the interaction integral does not vanish, but leads to terms representing the transfer of energy and momentum between the two components. Alternatively these equations may be combined and as in Section **1**, the mean velocity may be defined as $\varrho\boldsymbol{V}=(\varrho_++\varrho_-)\boldsymbol{V}=\varrho_+\boldsymbol{V}_++\varrho_-\boldsymbol{V}_-$, and $\boldsymbol{p}$, T etc. defined relative to V, whereupon

(I.2.4) $$\frac{\partial\varrho}{\partial t}+\operatorname{div}\varrho\boldsymbol{V}=0\,,$$

(I.2.5) $$\varrho\frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t}+\nabla\cdot\boldsymbol{p}-Q(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B})-\boldsymbol{j}\times\boldsymbol{B}=0\,,$$

and

(I.2.6) $$\frac{\mathrm{D}U}{\mathrm{D}t}+U\operatorname{div}\boldsymbol{V}+\boldsymbol{p}:\nabla\boldsymbol{V}+\operatorname{div}\boldsymbol{q}-\boldsymbol{j}\cdot\boldsymbol{E}=0\,.$$

To illustrate important methods used in solving the B.E. we shall first consider some simple representations of a simple gas, and only gradually approach the complexities of the ionized gas in a magnetic field.

3. – The normal solution: Hilbert's procedure.

To derive meaningful hydrodynamic equations it is useful to restrict attention first to those situations in which the rate of change of the distribution function is slow so that the collision frequency is much greater than any hydrodynamic frequency, *i.e.* if we introduce a macroscopic time scale, T, length scale, L, and a characteristic velocity, $V = LT^{-1}$, then, if the external forces are small, so that $T^{-2} < 1$, the L.H.S. of the B.E. scales as T^{-1}. We can also define a collision time by $\tau^{-1} \simeq n\sigma_0 V$, where σ_0 is a mean cross-section, whereupon the condition, collision frequency is much greater than hydrodynamic frequency, becomes $\tau/T = \varepsilon \ll 1$, and the B.E. may be written

$$I(f, f) = \varepsilon \left\{ \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{A}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}} \right\} f = \varepsilon\, \mathrm{D} f\,. \tag{I.3.1}$$

It now makes sense to seek a solution expanded in powers of ε:

$$f = f_0 + ef_1 + \dots\,,$$

which on being introduced into (I.3.1) reduces it to

$$I(f_0, f_0) = 0\,, \tag{I.3.2}$$

$$I(f_0, f_1) = Df_0\,. \tag{I.3.3}$$

The first equation here is satisfied by the locally Maxwellian distribution; *i.e.* by

$$f_0 = n(x, t) \left(\frac{m}{2\pi} kT \right)^{\frac{3}{2}} \exp\left[-\frac{m[\boldsymbol{v} - \boldsymbol{V}(x, t)]^2}{2kT(x, t)} \right], \tag{I.3.4}$$

where n, T, V are undetermined functions of x, t.

Hilbert observed that by writing $f_1 = f_0\phi$, (I.3.3) may be written

$$\int \mathrm{d}^3 v' f_0(\boldsymbol{v}) f_0(\boldsymbol{v}') |\boldsymbol{v} - \boldsymbol{v}'| \sigma(\boldsymbol{v} - \boldsymbol{v}', \theta) [\phi(\overline{\boldsymbol{v}}) + \phi(\overline{\boldsymbol{v}}') - \phi(\boldsymbol{v}) - \phi(\boldsymbol{v}')] = \mathrm{D} f_0\,,$$

i.e.

$$\int K(\boldsymbol{v}, \boldsymbol{v}') \phi \mathrm{d}^3 v' = \mathrm{D} f_0\,,$$

an integral equation in which conservation laws require that K should be symmetric in $\boldsymbol{v}$, $\boldsymbol{v}'$. This has the following interesting consequence: that a solution can be obtained only if Df_0 is orthogonal to the solution $h(\boldsymbol{v})$ of the homogeneous equation

$$\int K(\boldsymbol{v}, \boldsymbol{v}')\, h(\boldsymbol{v}')\, \mathrm{d}^3 v' = 0 ,$$

for

$$\int \mathrm{d}\boldsymbol{v} \int \mathrm{d}\boldsymbol{v}'\, h(\boldsymbol{v})\, K(\boldsymbol{v}, \boldsymbol{v}')\, \phi(\boldsymbol{v}') = 0 = \int \mathrm{d}\boldsymbol{v}\, h(\boldsymbol{v})\, \mathrm{D}f_0(\boldsymbol{v}) .$$

Since

$$\int K(v, v')\, \phi(v')\, \mathrm{d}v'$$

gives the rate of change of ϕ produced by collision, the solutions to the homogeneous equation are the collision invariants, m, $m\boldsymbol{v}$, $\frac{1}{2} m v^2$, and the constraints on Df_0 become the zero-order hydrodynamic equations. Since for a Maxwellian distribution $p_{ij} = nkT\delta_{ij}$, $\boldsymbol{q} = 0$, $U = \frac{3}{2} nkT$, these become

$$\frac{\partial n}{\partial t} + \boldsymbol{V} \cdot \nabla n + n \nabla \cdot \boldsymbol{V} = 0 , \tag{I.3.5}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \frac{1}{nm} [\nabla (nkT) - \boldsymbol{F}] = 0 , \tag{I.3.6}$$

$$\frac{\partial}{\partial t} \left(\frac{3}{2} nkT \right) + \boldsymbol{V} \cdot \nabla \left(\frac{3}{2} nkT \right) + \frac{5}{2} nkT \operatorname{div} \boldsymbol{V} = 0 . \tag{I.3.7}$$

Furthermore, since f_0 depends on x, and t through n, T and $\boldsymbol{V}$, we may write, with $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{V}$

$$\mathrm{D}f_0 = \left\{ \frac{1}{n} \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) n - \left(\frac{3}{2} - \frac{1}{2} \frac{mc^2}{kT} \right) \frac{1}{T} \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) T + \right.$$

$$\left. + \frac{m\boldsymbol{c}}{kT} \cdot \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{V} - \frac{m\boldsymbol{c}}{kT} \cdot \boldsymbol{A} \right\} f_0 .$$

The time derivatives may be eliminated with aid of (I.3.5)-(I.3.7) and

$$\mathrm{D}f_0 = \left\{ \left[\frac{m}{kT} \boldsymbol{c} \cdot (\boldsymbol{c} \cdot \nabla)) \boldsymbol{V} - \frac{1}{3} \frac{mc^2}{kT} \operatorname{div} \boldsymbol{V} \right] - \left[\frac{5}{2} - \frac{1}{2} \frac{mc^2}{kT} \right] \frac{1}{T} (\boldsymbol{c} \cdot \nabla) T \right\} f_0 . \tag{I.3.8}$$