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Environmental Stratified Flows

Roger Grimshaw

ENVIRONMENTAL STRATIFIED FLOWS

edited by

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Preface

The dynamics of flows in density-stratified fluids has been and remains now an important topic for scientific enquiry. Such flows arise in many contexts, ranging from industrial settings to the oceanic and atmospheric environments. It is the latter topic which is the focus of this book. Both the ocean and atmosphere are characterised by the basic vertical density stratification, and this feature can affect the dynamics on all scales ranging from the micro-scale to the planetary scale. The aim of this book is to provide a "state-of-the-art" account of stratified flows as they are relevant to the ocean and atmosphere with a primary focus on meso-scale phenomena; that is, on phenomena whose time and space scales are such that the density stratification is a dominant effect, so that frictional and diffusive effects on the one hand and the effects of the earth's rotation on the other hand can be regarded as of less importance. This in turn leads to an emphasis on internal waves.

The first three chapters deal with oceanic and atmospheric internal solitary waves, now recognised to be a highly significant component of the dynamics of the coastal ocean on the one hand, and the atmospheric boundary layer on the other hand. In the first chapter *Roger Grimshaw* reviews current theoretical models of oceanic and atmospheric internal solitary waves, emphasising the pivotal role of model evolution equations of the Korteweg-de Vries type. Then, in the second chapter it Peter Holloway, Efim Pelinovsky and *Tatiana Talipova* discuss both the theory and observations of oceanic internal solitary waves, while in the third chapter *Jim Rottman* and *Roger Grimshaw* do likewise for atmospheric solitary waves. The closely related topic of gravity currents and internal bores is then reviewed in the fourth chapter by *Jim Rottman* and *Paul Linden.*

Then, in chapter five *Ron Smith* reviews theoretical models for internal waves generated by flow over mountains. Inevitably density-stratified flows can be turbulent and this issue is addressed in chapter six by it Joe Fernando. In density-stratified flows as elswhere in fluid mechanics there is much to be learned from laboratory studies and so in chapter seven *Don Boyer* and It Andjelka Srdic-Mitrovic review laboratory studies of the flow of stratified fluids past obstacles. Then in chapter eight *Larry Redekopp* provides a comprehensive review and tutorial of the stability theory of stratified shear flows.

ROGER GRIMSHAW

Contents

Chapter 1

INTERNAL SOLITARY WAVES

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Abstract The basic theory of internal solitary waves is developed, with the main emphasis on environmental situations, such as the many occurrences of such waves in shallow coastal seas and in the atmospheric boundary layer. Commencing with the equations of motion for an inviscid, incompressible density-stratified fluid, we describe asymptotic reductions to model long-wave equations, such as the well-known Korteweg-de Vries equation. We then describe various solitary wave solutions, and propose a variable-coefficient extended Korteweg-de Vries equations as an appropriate evolution equation to describe internal solitary waves in environmental situations, when the effects of a variable background and dissipation need to be taken into account.

1. INTRODUCTION

Solitary waves are finite-amplitude waves of permanent form which owe their existence to a balance between nonlinear wave-steepening processes and linear wave dispersion. Typically, they consist of a single isolated wave of elevation, or depression, depending on the background state, whose speed is an increasing function of the amplitude. They are ubiquitous, and in particular internal solitary waves are a commonly occuring feature in the stratified flows of coastal seas, fjords and lakes (see, for instance, the reviews by Apel (1980, 1995) and Ostrovsky and Stepanyants (1989), as well as Chapter 2 of this monograph), and in the atmospheric boundary layer (see, for instance, the reviews by Smith (1988) and Christie (1989), as well as Chapter 3 of this monograph). Moreover, solitary waves are notable, not only because of their widespread occurrence, but also because they can be described by certain generic model equations which are either integrable, or close to integrability. The most notable example in this context is the now famous Korteweg-de Vries equation, which will figure prominently in the sequel.

In this Chapter, our aim is to develop appropriate model equations to describe internal solitary waves, and indicate, albeit rather briefly, some of their more salient properties. In the next section we will demonstrate how canonical model equations can be systematically derived from the complete fluid equations of motion for an inviscid, incompressible, density-stratified, fluid, with boundary conditions appropriate to an oceanic situation. The modifications necessary to model the lower atmosphere are readily made, and will be taken up in Chapter 3 of this monograph. Our main focus is on the Korteweg-de Vries equation, but importantly, in order to account for the large amplitudes sometimes observed, we extend this model to the extended Korteweg-de Vries equation which contains both quadratic and cubic nonlinearity. We shall describe the solitary wave solutions of these equations before turning, in the third section, to the modifications necessary to incorporate the effects of a variable background environment and dissipative processes. The outcome is a variable-coefficient extended Korteweg-de Vries equation. In general this model equation needs to be solved numerically, but to give some insight into the nature of the solutions, we describe a particular class of asymptotic solutions describing a slowly-varying solitary wave. This section also contains a brief account of unsteady "undular bores", insofar as they can be described by the Korteweg-de Vries equation. The Chapter concludes with a discussion of some outstanding issues.

2. LONG WAVE MODELS

2.1 Governing Equations

We shall begin by considering an inviscid, incompressible fluid which is bounded above by a free surface and below by a flat rigid boundary (see Figure 1). Initially we shall suppose that the flow is two-dimensional and can be described by the spatial coordinates (x, z) where x is horizontal and z is vertical. This configuration is appropriate for the modelling of internal solitary waves in coastal seas, and to some extent in straits, fjords or lakes provided that the effect of lateral boundaries can be ignored. The extensions to this basic model needed to incorporate these lateral effects, the effects of a horizontally variable background state, and various dissipative processes, will be described later in this chapter. The modifications needed to adapt this model to describe atmospheric solitary waves will be developed in Chapter 3.

Here, in the basic state the fluid has density $\rho_0(z)$, a corresponding pressure $p_0(z)$ such that $p_{0z} = -g\rho_0$ describes the basic hydrostatic equilibrium, and a horizontal shear flow in the x−direction. Then, in standard notation, the equations of motion relative to this basic state are

$$\rho_0(u_t+u_0u_x+wu_{0z})+p_x=-(\rho_0+\rho)(uu_x+wu_z)-\rho(u_t+u_0u_x+wu_{0z}), \tag{2.1a}$$

$$p_z+g\rho=-(\rho_0+\rho)(w_t+u_0w_x+uw_x+ww_z) \tag{2.1b}$$

$$g(\rho_t+u_0\rho_x)-\rho_0N^2w=-g(u\rho_x+w\rho_z) \tag{2.1c}$$

$$u_x+w_z=0 \tag{2.1d}$$

Here (u_0+u, w) are the velocity components in the (x, z) directions, $\rho_0+\rho$ is the density, p_0+p is the pressure and t is time. $N(z)$ is the bouyancy frequency, defined by

$$\rho_0N^2=-g\rho_{0z} \tag{2.2}$$

The boundary conditions are

$$w=0, \quad \text{at} \quad z=-h \tag{2.3a}$$

$$p_0+p=0, \quad \text{at} \quad z=\eta, \tag{2.3b}$$

and

$$\eta_t+u_0\eta_x+u\eta_x=w, \quad \text{at} \quad z=\eta. \tag{2.3c}$$

Here, the fluid has undisturbed constant depth h, and η is the displacement of the free surface from its undisturbed position $z=0$. Note that the effect of the earth's rotation has been neglected at this stage.

To describe internal solitary waves we seek solutions whose horizontal length scales are much greater than h, and whose time scales are much greater than N^{-1}. We shall also assume that the waves have small amplitude. Then the dominant balance is obtained by equating to zero the terms on the left-hand side of (2.1a-d); together with the linearization of the free surface boundary conditions we then obtain the set of equations describing linear long wave theory. To proceed it is useful to use the vertical particle displacement ζ as the primary dependent variable. It is defined by

$$\zeta_t+u_0\zeta_x+u\zeta_x+w\zeta_z=w. \tag{2.4}$$

Note that it then follows that the perturbation density field is given by $\rho=\rho_0(z-\zeta)-\rho_0(z)\approx\rho_0N^2\zeta$ as $\zeta\longrightarrow 0$, where we have assumed that as $x\rightarrow-\infty$, the density field relaxes to its basic state. The isopycnal surfaces (i.e. $\rho_0+\rho=$ constant) are then given by $z=z_0+\zeta$ where z_0 is the level as $x\rightarrow-\infty$. In terms of ζ, the kinematic boundary condition (2.3c) becomes simply $\zeta=\eta$ at $z=\eta$.

Linear long wave theory is now obtained by omitting the right-hand side of equations (2.1a-d), and simultaneously linearising boundary conditions (2.3b,c). Solutions are sought in the form

$$\zeta = A(x - ct)\phi(z), \tag{2.5}$$

while the remaining dependent variables are then given by

$$u = (c - u_0)A(x - ct)\phi_z(z), \tag{2.6a}$$

$$P = \rho_0(c - u_0)^2 A(x - ct)\phi_z(z), \tag{2.6b}$$

and

$$\rho = \rho_0 N^2 A(x - ct)\phi(z). \tag{2.6c}$$

Here c is the linear long wave speed, and the modal functions $\phi(z)$ are defined by the boundary-value problem,

$$\{\rho_0(c - u_0)^2\phi_z\}_z + \rho_0 N^2\phi = 0, \quad \text{in} \quad -h < z < 0, \tag{2.7a}$$

$$\phi = 0 \quad \text{at} \quad z = -h, \tag{2.7b}$$

$$\text{and} \qquad (c - u_0)^2\phi_z = g\phi \quad \text{at} \quad z = 0. \tag{2.7c}$$

Typically, the boundary-value problem (2.7a-c) defines an infinite sequence of modes, $\phi_n^{\pm}(z)$, $n = 0, 1, 2, \ldots$, with corresponding speeds $c_n^{\pm}$. Here, the superscript "±" indicates waves with $c_n^+ > u_M = \max u_0(z)$ and $c_n^- < u_M = \min u_0(z)$ respectively. We shall confine our attention to these regular modes, and consider only stable shear flows. Nevertheless, we note that there may also exist singular modes with $u_m < c < u_M$ for which an analogous theory can be developed (Maslowe and Redekopp, 1980). Note that it is useful to let $n = 0$ denote the surface gravity waves for which c scales with $\sqrt{gh}$, and then $n = 1, 2, 3, \ldots$ denotes the internal gravity waves for which c scales with Nh. In general, the boundary-value problem (2.7a-c) is readily solved numerically. Typically, $\phi_n^{\pm}(z)$, $n = 1, 2, 3, \ldots$, have n extremal points in the interior of the fluid, and vanish near $z = 0$ (and, of course, also at $z = -h$)

2.2 Time Evolution

It can now be shown that, within the context of linear long wave theory, any localised initial disturbance will evolve into a set of outwardly propogating modes, each given by an expression of the form (2.5). Indeed, it can be shown that the solution of the linearised long wave equations is given asymptotically by

$$\zeta \sim \sum_{n=0}^{\infty} A_n^{\pm}(x - c_n^{\pm}t)\phi_n^{\pm}(z), \quad \text{as} \quad t \to \infty. \tag{2.8}$$

Here the amplitudes $A_n^{\pm}(x)$ are determined in terms of the initial conditions,

$$\zeta = \zeta^{(0)}(x,z), \quad u = u^{(0)}(x,z), \quad \text{at} \quad t = 0, \tag{2.9}$$

by the integral expressions,

$$I_n^{\pm} A_n^{\pm}(x) = \int_{-h}^{0} \rho_0 \{(c_n^{\pm} - u_0)\zeta_z^{(0)} + u^{(0)} + u_{0z}\zeta^{(0)}\}\phi_{nz}^{\pm}\, dz, \tag{2.10a}$$

where

$$I_n^{\pm} = 2\int_{-h}^{0} \rho_0 (c_n^{\pm} - u_0)\phi_{nz}^{\pm 2} dz \tag{2.10b}$$

Assuming thats the speeds $c_n^{\pm}$ of each mode are sufficiently distinct, it is sufficient for large times to consider just a single mode. Henceforth, we shall omit the indices and assume that the mode has speed c, amplitude A and modal function $\phi(z)$. Then, as time increases, we expect the hitherto neglected nonlinear terms to have an effect, and to cause wave steepening. However, this is opposed by the terms representing linear wave dispersion, also neglected in the linear long wave theory. We expect a balance between these two effects to emerge as time increases. It is now well-known that the outcome is the Korteweg-de Vries (KdV) equation, or a related equation, for the wave amplitude.

The formal derivation of the evolution equation requires the introduction of the small parameters, α and ϵ, respectively characterising the wave amplitude and dispersion. A KdV balance requires $\alpha = \epsilon^2$, with a corresponding timescale of ϵ^{-3}. The asymptotic analysis required is well understood (e.g. Benney (1996), Lee and Beardsley (1974), Ostrovsky (1978), Maslowe and Redekopp (1980), Grimshaw (1981a), Tung *et al* (1981)), so we shall give only a brief outline here. We introduce the scaled variables

$$\tau = \epsilon\alpha t\,, \quad \theta = \epsilon(x - ct) \tag{2.11}$$

and then let

$$\zeta = \alpha A(\theta,\tau)\phi(z) + \alpha^2\zeta_2 + \ldots, \tag{2.12}$$

with similar expressions analgous to (2.6a-c) for the other dependent variables. At leading order, we get the linear long wave theory for the modal function $\phi(z)$ and the speed c, defined by (2.7a-c). Note that since the modal equation is homogeneous, we are free to impose a normalization condition on $\phi(z)$. A commonly used condition is that $\phi(z_m) = 1$ where $|\phi(z)|$ achieves a maximum value at $z = z_m$. In this case the amplitude αA is uniquely defined as the amplitude of ζ (to $0(\alpha)$) at the depth z_m. Then, at the next order, we obtain the equation for ζ_2,

$$\{\rho_0(c - u_0)^2\zeta_{2\theta z}\}_z + \rho_0 N^2\zeta_{2\theta} = M_2, \quad \text{in} \quad -h < z < 0, \tag{2.13a}$$

$$\zeta_{2\theta} = 0, \quad \text{at} \quad z = -h, \tag{2.13b}$$

$$\rho_0(c-u_0)^2\zeta_{2\theta z} - \rho_0 g\zeta_{2\theta} = N_2, \quad \text{at} \quad z = 0. \tag{2.13c}$$

Here the inhomogeneous terms M_2, N_2 are known in terms of $A(\theta,\tau)$ and $\phi(z)$, and are given by

$$M_2 = 2\{\rho_0(c-u_0)\phi_z\}_z A_\tau + 3\{\rho_0(c-u_0)^2\phi_z^2\}_z AA_\theta - \rho_0(c-u_0)^2\phi A_{\theta\theta\theta}, \tag{2.14a}$$

$$N_2 = 2\{\rho_0(c-u_0)\phi_z\}A_\tau + 3\{\rho_0(c-u_0)^2\phi_z^2\}AA_\theta. \tag{2.14b}$$

Note that the left-hand side of the equations (2.13a-c) is identical to the equations defining the modal function (i.e. (2.7a-c)), and hence can be solved only if a certain compatibility condition is satisfied. To obtain this compatibility condition, we first note that a formal solution of (2.13a) which satisfies the boundary condition (2.13b) is

$$\zeta_{2\theta} = A_{2\theta}\phi + \phi\int_{-h}^{z}\frac{M_2\psi}{W}\,dz - \psi\int_{-h}^{z}\frac{M_2\phi}{W}\,dz, \tag{2.15a}$$

where

$$W = \rho_0(c-u_0)^2\{\phi_z\psi - \psi_z\phi\}. \tag{2.15b}$$

Here $\psi(z)$ is a solution of the modal equation (2.7a) which is linearly independent of $\phi(z)$, and so, in particular $\psi(-h) \neq 0$. W (2.15b) is the Wronskian of these two solutions, and is a constant independent of z. Indeed, the expression (2.15b) can then be used to obtain ψ explicitly in terms of ϕ. The homogeneous part $A_{2\theta}\phi$ of the expression (2.15a) for $\zeta_{2\theta}$ introduces the second-order amplitude $A_2(\theta,\tau)$. Next, we insist that the expression (2.15a) for $\zeta_{2\theta}$ should satisfy the boundary condition (2.13c). The result is the compatibility condition

$$\int_{-h}^{0} M_2\phi\,dz = [N_2\phi]_{z=0} \tag{2.16}$$

Note that the amplitude A_2 is left undetermined at this stage.

Substituting the expressions (2.14a,b) into (2.16) we obtain the required evolution equation for A, namely the KdV equation

$$A_\tau + \mu AA_\theta + \lambda A_{\theta\theta\theta} = 0. \tag{2.17}$$

Here, the coefficients μ and λ are given by

$$I\mu = 3\int_{-h}^{0}\rho_0(c-u_0)^2\phi_z^3\,dz, \tag{2.18a}$$

$$I\lambda = \int_{-h}^{0}\rho_0(c-u_0)^2\phi^2\,dz, \tag{2.18b}$$

where

$$I = 2\int_{-h}^{0} \rho_0(c - u_0)\phi_z^2\, dz. \tag{2.18c}$$

Note that here I is just $I_n^{\pm}$ (2.10b) with the subscript and superscript omitted. Confining attention to waves propagating to the right, so that $c > u_M = \max u_0(z)$, we see that I and λ are always positive. Further, if we normalise the first internal modal function $\phi(z)$ so that it is positive at its extremal point, then it is readily shown that for the usual situation of a near-surface pycnocline, μ is negative for this first internal mode. However, in general μ can take either sign, and in some special situations may even be zero. Explicit evaluation of the coefficients μ and λ requires knowledge of the modal function, and hence they are usually evaluated numerically.

Proceeding to the next highest order will yield an equation set analogous to (2.13a-c) for ζ_3, whose compatibility condition then determines an evolution equation for the second-order amplitude A_2. We shall not give details here, but note that using the transformation $A + \alpha A_2 \to A$, and then combining the KdV equation (2.17) with the evolution equation for A_2 will lead to a higher-order KdV equation for A, in which the right-hand side of the KdV equation (2.17) contains terms proportional to $\alpha A_{\theta\theta\theta\theta\theta}$, $\alpha AA_{\theta\theta\theta}$, $\alpha A_\theta A_{\theta\theta}$ and $\alpha A^2 A_\theta$ (see for instance, Gear and Grimshaw (1983), Lamb and Yan (1996), and Grimshaw *et al* (1997)).

A particularly impotant special case of the higher-order KdV equation arises when the nonlinear coefficient μ (2.18a) in the KdV equation is close to zero. In this situation, the cubic nonlinear term in the higher-order KdV equation is the most important higher-order term. The KdV equation (2.17) may then be replaced by the extended KdV equation,

$$A_\tau + \mu AA_\theta + \alpha\nu A^2 A_\theta + \lambda A_{\theta\theta\theta} = 0. \tag{2.19}$$

For $\mu \approx 0$, a rescaling is needed and the optimal choice is to assume that μ is $0(\epsilon)$, and then replace A with A/ϵ. In effect the amplitude parameter is ϵ in place of ϵ^2. The coefficient ν of the cubic nonlinear term is given by

$$I\nu = 3\int_{-h}^{0} \rho_0(c-u_0)^2\phi_z^2\{3\chi_z - 2\phi_z^2\}dz + \tag{2.20a}$$
$$\mu\int_{-h}^{0} \rho_0(c-u)0)\phi_z\{5\phi_z^2 - 4\chi_z\}dz - \mu^2\int_{-h}^{0}\rho_0\phi_z^2 dz,$$

where

$$\{\rho_0(c-u_0)^2\chi_z\}_z + \rho_0 N^2\chi = \left\{\frac{3}{2}\rho_0(c-u_0)^2\phi_z^2\right\}_z - \mu\{\rho_0(c-u_0)\phi_z\}_z, \tag{2.20b}$$

$$\chi = 0 \quad \text{at} \quad z = -h, \tag{2.20c}$$

and

$$\rho_0(c-u_0)^2\chi_z - \rho_0 g\chi = \frac{3}{2}\rho_0(c-u_0)^2\phi_z^2 - \mu\rho_0(c-u_0)\phi_z \text{ at } z=0. \tag{2.20d}$$

Note here that, although the terms with coefficients μ or μ^2 can be omitted in the asymptotic limit $\epsilon \to 0$, it is useful in practice to retain them so that this expression for ν remains valid even when μ is not small.

The function $\chi(z)$ is determined from the equation set (2.20b-d), which can be recognised as an inhomogeneous form of the modal equation set (2.7a-c). Indeed, it is readily seen from (2.13a-c) that

$$\zeta_2 = A_2\phi + A^2\chi + A_{\theta\theta}\hat{\chi}, \tag{2.21}$$

where the function $\hat{\chi}(z)$ also satisfies an inhomogeneous form of the modal eqution, analogous to (2.20b-d), but with the right-hand side of (2.20b) replaced with $-\rho_0(c-u_0)^2\phi - 2\lambda\{\rho_0(c-u_0)^2\phi_z\}_z$, and the right-hand side of (2.20d) replaced by $-2\lambda\rho_0(c-u_0)\phi_z$. Of course here, we must use the compatibility condition (2.16), which is just the KdV equation (2.17), to eliminate A_τ from M_2 and N_2. But now we see that the equation set (2.20b-d) does not define $\chi(z)$ uniquely, and hence ν (2.20a) is not unique either. Indeed we can always add a term $\gamma\phi(z)$ to χ, which has the effect of adding a term $\gamma\mu$ to ν. But this is just equivalent to the transformation $A_2 \to A_2 + \gamma A^2$, or $A \to A + \alpha\gamma A^2$, and it is then readily verified that this will asymptotically transform (2.19) into itself with ν replaced by $\nu + \alpha\mu$. Thus, the lack of uniqueness in ν is related to a lack of uniqueness in A_2, or equivalently in ζ_2. The remedy is that we are free to impose an extra condition on η. For instance, if we suppose that $\chi_z = 0$ at $z = -h$, then it follows that $\chi = \chi_p$ say, where

$$\chi_p = \phi\int_{-h}^{z}\frac{f_2\psi}{W}dz - \psi\int_{-h}^{z}\frac{f_2\phi}{W}dz \tag{2.22}$$

where f_2 is the right-hand side of (2.20b), and we recall that ψ is defined by (2.15b). The expression (2.22) is readily evaluated numerically, and is consequently recommended as a standard for the calculation of ν. However, if an alternative condition is required, then it can readily be found by adding a term $\gamma\phi$ to χ_p, and using the new condition to determine γ. For instance, it is sometimes useful to require that χ (and also $\hat{\chi}$) vanish at z_m, where we recall that $\phi(z_m) = 1$, and $z = z_m$ locates the maximum value of $|\phi(z)|$. In this case we simply have that $\gamma = -\chi_p(z_m)$, and then $\chi = \chi_p + \gamma\phi$. Thus, $\zeta = \alpha A\phi + \alpha^2\zeta_2 + \ldots$, and the amplitude

$\alpha A + \alpha^2 A_2$ is uniquely defined as the amplitude of ζ (to $0(\lambda^2)$) at the depth z_m.

In some atmospheric and oceanic applications, the depth h is not necessarily small relative to the horizontal length scale of the solitary wave, but nevertheless the density stratification is effectively confined to a thin layer of depth h_1, which is much shorter that the horizontal length scales. In this case, a different theory is needed, and was first developed by Benjamin (1967) and Davis and Acrivos (1967). Several variants are possible, so, to be specific, we shall describe an oceanic case when $\rho_0(z)$, and $u_0(z)$ vary only in a near-surface layer of depth h_1, below which $\rho_0(z) = \rho_\infty$ (a constant) and $u_0(z) = 0$, while the ocean bottom is now given by $z = -H/\epsilon$ (i.e. $H = \epsilon h$). The modal function is again defined by (2.7a,c) but the bottom boundary condition (2.7b) is now replaced by a matching condition that $\phi_z \to 0$ as $z \to -\infty$. To derive the evolution equation, we again use the asymptotic expansion (2.12) but now with $\alpha = \epsilon$ and restricted to the near-surface layer. This expansion is matched to an appropriate solution in the deep-fluid region where Laplace's equation holds at leading order. The outcome is the intermediate long-wave (ILW) equation (Kubota *et al* (1978), Maslowe and Redekopp (1980), Grimshaw (1981a), Tung *at al* (1981),

$$A_\tau + \mu A A_\theta + \delta \mathcal{L}(A_\theta)) = 0, \tag{2.23a}$$

where

$$\mathcal{L}(A) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} k \coth kH \exp(ik\theta)\mathcal{F}(A)dk, \tag{2.23b}$$

and

$$\mathcal{F}(A) = \int_{-\infty}^{\infty} A \exp(-ik\theta))d\theta \tag{2.23c}$$

Here the nonlinear coefficient μ is again given by (2.18a) with $-h$ now replaced by $-\infty$, while the dispersive coefficient δ is defined by $I\delta = (\rho_0 c^2 \phi^2)_{z \to -\infty}$. In the limit $H \to \infty$, $k \coth kH \to |k|$ on the integrand of (2.21b) and (2.21a) becomes the Benjamin-Ono (BO) equation. In the opposite limit $H \to 0$, (2.23a) reduces to a KdV equation.

An important variant of the ILW equation (2.23a) arises when it is supposed that the deep ocean is infinitely deep ($H \to \infty$) and weakly stratified, with a constant buoyancy frequency ϵN_0. Then the operator $\mathcal{L}(A)$ in (2.21a) is replaced by (Maslowe and Redekopp (1980), Grimshaw (1981b))

$$\mathcal{L}_m(A) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}(k^2 - m^2)^{\frac{1}{2}} \exp(ik\theta)\mathcal{F}(A)dk, \tag{2.24}$$

where $m = N_0/c$. Now internal gravity waves can propagate vertically in the deep fluid region, and to ensure that these waves are outgoing, a radiation condition is needed. Thus $(k^2 - m^2)^{\frac{1}{2}}$ is either real and positive for $k^2 > m^2$, or $i$sign $k(m^2 - k^2)^{\frac{1}{2}}$ for $k^2 < m^2$. As $m \to 0$, (2.24) becomes the BO equation.

2.3 Solitary Waves

Each of the evolution equations (viz. the KdV equation (2.17), the extended KdV equation (2.19) and the ILW equation (2.23a)) are exactly integrable (see, for instance, Ablowitz and Segur (1981), or Dodd *et al* (1982)), with the consequence that the initial-value problem with a localised initial condition is exactly solvable. But note that the variant (2.24) is not integrable. The most important implication of this integrability from the perspective of this monograph is that an arbitrary initial disturbance will evolve into a finite number (N) of solitary waves (called solitons in this context) and an oscillatory decaying tail. This, together with the robust stability properties of solitary waves, explains why internal solitary waves are so commonly observed. Note that because solitary waves typically have speeds which increase with the wave amplitude, the N waves are rank-ordered by amplitude as $t \to \infty$. Also, to produce solitary waves at all, the initial disturbance should have the correct polarity (e.g. $\mu \int_{-\infty}^{\infty} A(\theta, 0) d\theta > 0$ for the case of the KdV equation (2.17)). A typical solution of the KdV equation showing the generation of solitary waves is shown in Figure 2. Note that, in applications the initial condition $A(\theta, 0)$ for the evolution equation is found by first solving the linear long wave equations, and then identifying the mode of interest. Thus $A(\theta, 0)$ is given by (2.10a) in terms of the actual initial conditions (2.9).

It follows from the proceeding discussion that in describing the solution of the evolution equations, the most important step is to determine the solitary wave solution. For the KdV equation (2.17) this is given by

$$A = a\,\mathrm{sech}^2\beta(\theta - V\tau) \qquad (2.25a)$$

where

$$V = \frac{1}{3}\mu a = 4\lambda\beta^2. \qquad (2.25b)$$

Note that the speed V is for the phase variable θ, and the actual total speed is $c + \alpha V$. Since the dispersion coefficient λ is always positive for right-going waves, it follows that these solitary waves are always supercritical ($V > 0$), and are waves of elevation or depression according as $\mu \gtrless 0$. We also see that β^{-1} is proportional to $|a|^{-\frac{1}{2}}$, and hence the larger waves are not only faster, but narrower.