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1

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by C. Chevalley

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PREFACE

The theory of exterior algebras was introduced by Grassmann to study algebraically geometric problems on the linear varieties in a projective space. But this theory has been forgotten for a long time; E. Cartan discovered it again and applied it to the study of differentials and their multiple integrals over a differentiable or analytic variety. For this reason, the theory of exterior algebras will be interesting not only for algebraists but also for analysts.

In these lectures we shall present a more general algebra called "Clifford algebra" associated to a quadratic form. If the quadratic form reduces to 0, the Clifford algebra reduces to "exterior algebra".

The applications of the theory of exterior algebras are very wide, e.g.: theory of determinants, representation of linear variety in a projective space using Plücker coordinates, and the theory of differential forms and their applications to many branches of analysis. But I am sorry not to be able to describe them in detail because of the restriction of time.

June, 1954 C. Chevalley

CONVENTIONS

Throughout these lectures, we mean by a ring a ring with unit element 1 (or $1'$ as the case may be), and also by a homomorphism of such rings a homomorphism which maps unit upon unit. A will always denote a ring which is quite arbitrary in Chap. I, and assumed to be commutative in Chap. II and the subsequent chapters.

By a *module* over A, we invariably mean a unitary module. Thus a module over A is a set M such that

1) M has a structure of an additive group,
2) for every $\alpha \in A$ and $x \in M$, an element $\alpha x \in M$ called *scalar multiple* is defined and we have
   i) $\alpha(x+y)=\alpha x+\alpha y$,
   ii) $(\alpha+\beta)x=\alpha x+\beta x$,
   iii) $\alpha(\beta x)=(\alpha\beta)x$,
   iv) $1\cdot x=x$.

A map of a module over A into a module over A is called *linear*, if it is a homomorphism of the underlying additive groups which commutes with every scalar multiplication by every element of A.

An *algebra* E over A means a module over A with an associative multiplication which makes E a ring satisfying

$$\alpha(xy)=(\alpha x)y=x(\alpha y) \qquad (x, y \in E;\ \alpha \in A).$$

A homomorphism of algebras will always mean a ring homomorphism which maps unit upon unit. An ideal of an algebra means always a *two-sided ideal*. A subset S of an algebra is called *a set of generators* of E if E is the smallest subalgebra containing S and *the unit* 1 of E.

In dealing with modules or algebras over A, any element of the basic ring A is often called a *scalar*. In the case of algebras, any element of the subalgebra $A\cdot 1$ is called a scalar; a scalar clearly commutes with every element of the algebra.

CHAPTER I. GRADED ALGEBRAS.

§ 1. **Free algebras.** The first basic type of algebras we want to consider is the free algebra. Let E be an algebra over A generated by a given set of generators $(x_i)_{i\in I}$ (I: any set of indices). Let $\sigma=(i_1,\cdots,i_h)$ be a finite sequence of elements of I and put $y_\sigma=x_{i_1}\cdots x_{i_h}$. The number h is called the *length* of σ. Among the "finite sequences" we always admit the empty sequence σ_0, whose length is 0, i.e., a sequence with no term, and we put $y_{\sigma_0}=1$. We define the composition of two finite sequences $\sigma=(i_1,\cdots,i_h)$ and $\sigma'=(j_1,\cdots,j_k)$ by $\sigma\sigma'=(i_1,\cdots,i_h,j_1,\cdots,j_k)$. For σ_0, we define $\sigma_0\sigma=\sigma\sigma_0=\sigma$, i.e., σ_0 is the unit for this composition. Evidently this composition is associative: $(\sigma\sigma')\,\sigma''=\sigma(\sigma'\sigma'')$, and we have $y_{\sigma\sigma'}=y_\sigma y_{\sigma'}$.

THEOREM 1.1. *Every element of E is a linear combination of the y_σ's, σ running over all finite sequences of elements of I.*

PROOF. Denote by E_1 the module spanned by all the y_σ's. We shall show $E=E_1$. First we prove:

LEMMA 1.1. *E_1 is closed under multiplication.*

PROOF. Let z, z' be two elements of E_1 and put

$$z=\sum_\sigma a_\sigma y_\sigma\,,\qquad z'=\sum_\sigma a'_\sigma y_\sigma.$$

Though these two sums seem apparently infinite, we have in fact $a_\sigma=0$ and $a'_\sigma=0$ except for a finite number of σ's. Then we have

$$zz'=\sum_{\sigma,\sigma'} a_\sigma a'_{\sigma'} y_{\sigma\sigma'}\,,\qquad y_{\sigma\sigma'}\in E_1\,;$$

the sum being finite, we have $zz'\in E_1$.

Now we return to the proof of Theorem 1.1. The module E_1 is thus a subalgebra of E, and if $\sigma=(i)$, $y_\sigma=x_i$ and also $y_{\sigma_0}=1$. Therefore E_1, containing the set of generators (x_i) and 1, contains E itself, so that we obtain $E=E_1$, which proves the theorem.

DEFINITION 1.1. *If the y_σ's are linearly independent over A, then E is called a free algebra, and the set $(x_i)_{i\in I}$ is called a free system of generators of E.*

Existence and uniqueness of free algebras. We first prove the *uniqueness*. For this, we shall show a more precise condition called "universality". An algebra F over A with a system of generators $(x_i)_{i\in I}$ is called *universal*, if given any algebra E over A generated by a set of elements $(\xi_i)_{i\in I}$ indexed by the same set I, there is a unique homomorphism $\varphi: F\to E$ such that $\varphi(x_i)=\xi_i$ for all i.

THEOREM 1.2. *A free algebra F with its free system of generators is universal.*

PROOF. By definition, the set $\{y_\sigma = x_{i_1}\cdots x_{i_h}\}$ forms a base of F as a *module* over A. Thus there is a *linear* mapping $\varphi: F\to E$ such that

(1) $$\varphi(y_\sigma)=\xi_{i_1}\cdots\xi_{i_h} \quad \text{for every } \sigma=(i_1,\cdots,i_h).$$

If $\sigma=(i_1,\cdots,i_h)$, $\sigma'=(j_1,\cdots,j_k)$ are two finite sequences of I, we have

(2) $$\varphi(y_\sigma y_{\sigma'})=\varphi(y_{\sigma\sigma'})=\xi_{i_1}\cdots\xi_{i_h}\xi_{j_1}\cdots\xi_{j_k}=\varphi(y_\sigma)\varphi(y_{\sigma'}).$$

This proves that φ is not only linear, but also a homomorphism $F\to E$. Especially putting $\sigma=(i)$ resp. $\sigma=\sigma_0$, we have $\varphi(x_i)=\xi_i$ and $\varphi(1)=1$, which prove our assertion.

Remark that, in general, any homomorphism φ is uniquely determined when the values $\varphi(x_i)$ on a set of generators (x_i) are given.

COROLLARY. *The free algebra generated by $(x_i)_{i\in I}$ is unique under isomorphism. More precisely, let F, F' be two free algebras with free systems of generators $(x_i)_{i\in I}$, $(x'_{i'})_{i'\in I'}$ respectively, and let I and I' be equipotent. Then F and F' are isomorphic.*

PROOF. We may assume that $I=I'$. By Theorem 1.2, we have two homomorphisms

$$\varphi : F\to F' \qquad \text{such that } \varphi(x_i)=x'_i$$

and

$$\varphi' : F'\to F \qquad \text{such that } \varphi'(x'_i)=x_i$$

The composite mapping $\varphi'\circ\varphi$[1]: $F\to F'\to F$ maps each x_i to itself,

1) $\varphi'\circ\varphi$ is defined by $\varphi'\circ\varphi(x)=\varphi'(\varphi(x))$.

and by the uniqueness of homomorphism, $\varphi' \circ \varphi$ must be the identity in F. Similarly $\varphi \circ \varphi'$ is the identity in F'. Therefore φ is an isomorphism and $\varphi' = \varphi^{-1}$ which proves that F and F' are isomorphic to each other.

Now we shall prove the *existence* of a free algebra, having any given set $(x_i)_{i \in I}$ as its free system of generators. Let Σ be the set of all finite sequences of elements of I. From the theory of linear algebra, we may assume that there exists a module M over A with a base equipotent to Σ. Let $(y_\sigma)_{\sigma \in \Sigma}$ be the base of M; we introduce a structure of algebra into M. For this, we have only to define an associative multiplication for the elements of the base. We define it by

$$y_\sigma y_{\sigma'} = y_{\sigma\sigma'} .$$

Since the composition in Σ is associative, we have the associativity: $(y_\sigma y_{\sigma'}) y_{\sigma''} = y_\sigma (y_{\sigma'} y_{\sigma''})$. M is now a free algebra over A having the free system of generators $(x_i)_{i \in I}$.

§2. **Graded algebras.** Let F be the free algebra with the free system of generators $(x_i)_{i \in I}$, and put $y_\sigma = x_{i_1} \cdots x_{i_h}$ $(\sigma = (i_1, \cdots, i_h))$. We shall classify the elements y_σ by the length of σ.

Let F_h be the module spanned by the y_σ's, σ being of length h. Then F is the direct sum of $F_0, F_1, F_2, \cdots$ as a module:

(1) $$F = F_0 + F_1 + F_2 + \cdots + F_h + \cdots$$

and evidently

(2) $$F_h \cdot F_{h'} \subset F_{h+h'} ,$$

because the length of the composite $\sigma\sigma'$ of σ and σ' is equal to the sum of the lengths of σ and σ'.

The free algebra $F = F_0 + F_1 + \cdots + F_h + \cdots$ is a typical example of the following general notion of *graded algebras*.

DEFINITION 1.2. *Let Γ be an additive group. A Γ-graded algebra is an algebra E which is given together with a direct sum decomposition as a module*

(3) $$E = \sum_{\gamma \in \Gamma} E_\gamma$$

where the E_γ's are submodules of E, in such a way that

(4) $E_\gamma \cdot E_{\gamma'} \subset E_{\gamma+\gamma'}$, *i.e.*, $x \in E_\gamma$ *and* $x' \in E_{\gamma'}$ *imply* $xx' \in E_{\gamma+\gamma'}$.

By a homomorphism of Γ-graded algebra $E=\sum_{\gamma\in\Gamma} E_\gamma$ into another Γ-graded algebra $E'=\sum_{\gamma\in\Gamma} E'_\gamma$ is meant a homomorphism $\varphi: E \to E'$ of the algebras such that $\varphi(E_\gamma) \subset E'_\gamma$.

In a Γ-graded algebra $E=\sum E_\gamma$ an element belonging to E_γ is called *homogeneous* of degree γ. The zero element 0 of E is homogeneous of any degree, but each element of E other than 0 is homogeneous of at most one degree $\gamma \in \Gamma$. Any element x of E is uniquely decomposed into the sum of homogeneous elements

(5) $$x=\sum_{\gamma\in\Gamma} x_\gamma, \qquad x_\gamma \in E_\gamma,$$

where the x_γ's are 0 except for a finite number of γ's. Each x_γ in (5) is called the γ *component* of x.

LEMMA 1.2. *The unit 1 is always homogeneous of degree θ (θ: zero element of Γ).*

PROOF. Decompose 1 into the sum of its homogeneous components:

$$1=\sum_{\gamma\in\Gamma} e_\gamma, \qquad e_\gamma \in E_\gamma.$$

If $x_\beta \in E$ is homogeneous of degree $\beta \in \Gamma$, then we have

$$E_\beta \ni x_\beta = x_\beta \cdot 1 = \sum_\gamma x_\beta \cdot e_\gamma.$$

Since $x_\beta \cdot e_\gamma \in E_{\beta+\gamma}$, we must have $x_\beta \cdot e_\theta = x_\beta$ and $x_\beta \cdot e_\gamma = 0$ for all $\gamma \neq \theta$. This implies that e_θ is a right unit element for all homogeneous elements, and accordingly for all elements $x=\sum x_\gamma$ in E. Thus $e_\theta = 1$, and our assertion is proved.

COROLLARY. *Scalars are homogeneous of degree θ (θ: zero element of Γ).*

Among others, the following two special types of Γ-gradations are of much importance:

i) Γ-gradations where $\Gamma=Z$ is the additive group of integers. In this case, we say simply "graded" instead of "Z-graded".

ii) Γ-gradations where Γ is the group with two elements 0 and 1. In this case we write $E=E_+ + E_-$ in place of $E=E_0+E_1$, and E is called *semi-graded*.

A free algebra $F=F_0+F_1+\cdots+F_h+\cdots$ can be considered as a graded algebra with $F_h=\{0\}$ for all $h<0$.

REMARK. A Γ-graded algebra is not a special kind of algebras. In fact, any algebra may be considered as a Γ-graded algebra with degree θ for every element.

Homogeneous subalgebras.

DEFINITION 1.3. *A submodule M of a Γ-graded algebra $E=\sum E_\gamma$ is said to be homogeneous if the homogeneous components of any element of M still belong to M. This is equivalent to the condition that $M=\sum_\gamma (M \cap E_\gamma)$.*

THEOREM 1.3. *If a submodule M or an ideal $\mathfrak{A}$ of a Γ-graded algebra E is generated by*[2] *homogeneous elements, then it is homogeneous.*

PROOF. Let M be a submodule of E spanned by a set S of homogeneous elements and let M' be the set of elements of M whose homogeneous components belong to M. It is evident that $S \subset M' \subset M$, since S consists of homogeneous elements. We shall show that M' is a submodule. If $x=\sum x_\gamma$ and $x'=\sum x'_\gamma$ are in M', then $x \pm x' = \sum (x_\gamma \pm x'_\gamma)$, and $x_\gamma \pm x'_\gamma \in M$, so that we have $x \pm x' \in M'$. Also for $\alpha \in A$, we have similarly $\alpha x \in M'$. Thus M' being a submodule containing the generators S, we have $M' \supset M$, and so $M=M'$, which proves that M is homogeneous.

For the case of ideals, we take the ideal $\mathfrak{A}$ generated by a set S of homogeneous elements. $\mathfrak{A}$ is spanned, *as a module*, by all elements of the form $x s y$, where $x \in E$, $s \in S$ and $y \in E$. Putting $x=\sum x_\gamma$, $y=\sum y_\beta$, we have

2) The word "generated by" has somewhat different meaning for the cases of submodules and of ideals. In the former case, a submodule M is generated by S if every element of M is a linear combination of the elements of S, while in the latter case, an ideal $\mathfrak{A}$ is generated by S if $\mathfrak{A}$ is the smallest ideal containing the set S.

$$x\,s\,y=(\sum_{\gamma} x_{\gamma})s(\sum_{\beta} y_{\beta})=\sum_{\gamma,\beta} x_{\gamma}\,s\,y_{\beta}$$

and since $(x_{\gamma}\,s\,y_{\beta})$ is homogeneous, $\mathfrak{A}$ is also spanned by the elements $x_{\gamma}\,s\,y_{\beta}$ which are homogeneous. Thus $\mathfrak{A}$, being generated as a module by homogeneous elements, is homogeneous as was seen above.

Let $E=\sum E_{\gamma}$ be a Γ-graded algebra and $\mathfrak{A}$ a homogeneous ideal in E. We have the direct sum decomposition of $\mathfrak{A}$ into its homogeneous parts:

$$\mathfrak{A}=\sum_{\gamma}\mathfrak{A}_{\gamma}, \qquad \mathfrak{A}_{\gamma}=\mathfrak{A}\cap E_{\gamma}.$$

The quotient algebra $E/\mathfrak{A}$ has also the structure of *Γ-graded algebra*, because $E/\mathfrak{A}=\sum_{\gamma}(E_{\gamma}/\mathfrak{A}_{\gamma})$ (direct sum of submodules) and $(E_{\gamma}/\mathfrak{A}_{\gamma})\cdot(E_{\gamma'}/\mathfrak{A}_{\gamma'})\subset E_{\gamma+\gamma'}/\mathfrak{A}_{\gamma+\gamma'}$. Therefore $E/\mathfrak{A}$ is a Γ-graded algebra and $\sum_{\gamma}(E_{\gamma}/\mathfrak{A}_{\gamma})$ gives its homogeneous decomposition. The canonical homomorphism $\psi: E\to E/\mathfrak{A}$ is a homomorphism not only of algebras, but also of *Γ-graded algebras*.

§3. **Homogeneous linear mappings.**[3] Let E, E' be two Γ-graded algebras over the same ring A, and let λ be a linear mapping of E into E', i.e., a mapping $\lambda: E\to E'$ such that

$$\lambda(x+y)=\lambda(x)+\lambda(y), \quad \lambda(\alpha x)=\alpha\lambda(x)$$

$$\text{for every } x, y\in E;\quad \alpha\in A.$$

DEFINITION 1.4. *Let ν be any element of Γ; λ is called homogeneous of degree ν if $\lambda(E_{\gamma})\subset E'_{\gamma+\nu}$ for all $\gamma\in\Gamma$.*

Evidently, if $\lambda: E\to E'$ is homogeneous of degree ν and $\lambda': E'\to E''$ is homogeneous of degree ν', then $\lambda'\circ\lambda$ is homogeneous of degree $\nu+\nu'$.

A linear mapping $\lambda: E\to E'$ *can not always be decomposed* into a finite sum of homogeneous mappings as can be shown by a counter-example. But if the decomposition is *possible*, it is *unique*; it is sufficient to prove the following:

3) This notion can be defined not only for graded algebras, but also for "graded modules". But we shall restrict ourselves only to the case of graded algebras, because we use it only in this case.

LEMMA 1.3. *Let $\{\lambda_\nu\}_{\nu\in\Gamma}$ be a family of linear mappings $E\to E'$, in which each λ_ν is homogeneous of degree ν. If $\sum_\nu \lambda_\nu=0$ and $\lambda_\nu(x)=0$ (x: any element in E) except for a finite number of $\nu\in\Gamma$, then $\lambda_\nu=0$ for all $\nu\in\Gamma$.*

PROOF. For an element x_γ of E_γ, we have $\sum_\nu \lambda_\nu(x_\gamma)=0$, but since $\lambda_\nu(x_\gamma)\in E'_{\gamma+\nu}$ for each $\nu\in\Gamma$, we have $\lambda_\nu(x_\gamma)=0$ for all $\nu\in\Gamma$. For an arbitrary $x\in E$, let $x=\sum x_\gamma$ be the homogeneous decomposition of x, then $\lambda_\nu(x)=\sum_\gamma \lambda_\nu(x_\gamma)=0$, which proves that $\lambda_\nu=0$ ($\nu\in\Gamma$).

§ 4. **Associated gradations and the main involution.** Let Γ, $\tilde{\Gamma}$ be additive groups and let a homomorphism $\tau:\Gamma\to\tilde{\Gamma}$ be given. To any Γ-graded algebra $E=\sum_{\gamma\in\Gamma} E_\gamma$, we associate the following $\tilde{\Gamma}$-gradation of E. For each $\tilde{\gamma}\in\tilde{\Gamma}$, put

$$E_{\tilde{\gamma}}=\sum_{\gamma\in\tau^{-1}(\tilde{\gamma})} E_\gamma \quad (E_{\tilde{\gamma}}=\{0\} \text{ if } \tau^{-1}(\tilde{\gamma}) \text{ is empty}).$$

Then obviously $E=\sum_{\tilde{\gamma}\in\tilde{\Gamma}} E_{\tilde{\gamma}}$ and $E_{\tilde{\gamma}}\cdot E_{\tilde{\gamma}'}\subset E_{\tilde{\gamma}+\tilde{\gamma}'}$. In this way $E=\sum E_{\tilde{\gamma}}$ can be considered as a $\tilde{\Gamma}$-graded algebra.

DEFINITION 1.5. *The $\tilde{\Gamma}$-gradation $E=\sum_{\tilde{\gamma}\in\tilde{\Gamma}} E_{\tilde{\gamma}}$ is called the associated $\tilde{\Gamma}$-gradation of E, associated to the Γ-gradation $E=\sum_{\gamma\in\Gamma} E_\gamma$ (with respect to τ).*

We shall write E^τ instead of E if it is taken with the associated $\tilde{\Gamma}$-gradation rather than with the original Γ-gradation. Obviously, we have the

LEMMA 1.4. *Every homogeneous element, every homogeneous submodule, and every homogeneous ideal in E are also homogeneous in E^τ.*

In the special case where $\tilde{\Gamma}$ is the group consisting of two elements 0 and 1, and where τ is onto, we write $E^s=E^s_+ + E^s_-$ instead of $E^\tau=E_0+E_1$, and we call it the *associated semi-graded algebra* of E. In that case, the kernel $\tau^{-1}(0)\subset\Gamma$ is denoted by Γ_+, which is a subgroup of index 2, while $\tau^{-1}(1)\subset\Gamma$ is denoted by Γ_-, which is a coset of Γ by Γ_+ other than Γ_+. Remark that every subgroup of Γ of index 2 can be preassigned as Γ_+ in some unique associated semi-gradation. It may happen that Γ has a unique subgroup of index 2. If it is the case, then reference to the map τ can be omitted without any ambiguity. For example, to every graded (i.e., Z-graded) algebra $E=\sum_{h:\text{integer}} E_h$ is associated a unique semi-graded algebra $E^s=E^s_+ + E^s_-$, where $E^s_+=\sum_{h:\text{even}} E_h$, $E^s_-=\sum_{h:\text{odd}} E_h$. Clearly, if E is a semi-graded algebra, then its associated semi-gradation is identical with the original semi-gradation.

Main involution. Fixing a subgroup $\Gamma_+ \subset \Gamma$ of index 2, let $E=\sum_{\gamma\in\Gamma} E_\gamma$ be a Γ-graded algebra, and let $E^s=E^s_+ + E^s_-$ be the associated semi-gradation of E. Every element $x \in E$ can be decomposed uniquely into the sum of its E^s_+-component x_+ and its E^s_--component x_- : $x=x_+ + x_-$. If we define a map $J: E \to E$ by

$$J(x)=x_+ - x_- \quad (x=x_+ + x_- \in E),$$

then J is one-to-one and linear, preserves the degree in the Γ-gradation of E, maps unit upon unit, and is an involution (i.e., $J \circ J=$ identity). Moreover, J preserves the multiplication. In fact, let $x=x_+ + x_-$, $y=y_+ + y_-$ ($x_+, y_+ \in E^s_+$; $x_-, y_- \in E^s_-$). Then $(xy)_+ = x_+y_+ + x_-y_-$, $(xy)_- = x_-y_+ + x_+y_-$, and so we have

$$\begin{aligned} J(xy) &= (x_+y_+ + x_-y_-) - (x_-y_+ + x_+y_-) \\ &= (x_+ - x_-)(y_+ - y_-) = J(x)J(y) . \end{aligned}$$

Therefore, J is an involutive automorphism of the Γ-graded algebra E, which we call the *main involution* of E.

For convenience' sake, we define the symbolical power $J^\nu (\nu \in \Gamma)$ of the main involution as follows:

$$J^\nu = \begin{cases} J & \text{if} \quad \nu \in \Gamma_- \\ \text{identity} & \text{if} \quad \nu \in \Gamma_+ . \end{cases}$$

Also we define the power $(-1)^\nu$ $(\nu \in \Gamma)$ of the scalar (-1) of A as follows:

$$(-1)^\nu = \begin{cases} -1 & \text{if} \quad \nu \in \Gamma_- \\ 1 & \text{if} \quad \nu \in \Gamma_+ . \end{cases}$$

Then we have, just as in the case of usual powers, the following identities:

i) $J^{\nu} \circ J^{\nu'} = J^{\nu+\nu'}$

ii) $(-1)^{\nu}(-1)^{\nu'} = (-1)^{\nu+\nu'}$

iii) $(J^{\nu})^{\nu'} = (J^{\nu'})^{\nu}$

iv) $((-1)^{\nu})^{\nu'} = ((-1)^{\nu'})^{\nu}$

We shall denote iii) and iv) respectively by $J^{\nu\nu'}$ and by $(-1)^{\nu\nu'}$ for the sake of simplicity, though no product is defined, in general in Γ. Any power of the identity map is understood to be the identity map, and any power of 1 is understood to be 1.

If $x = \sum_{\gamma \in \Gamma} x_{\gamma}$ $(x_{\gamma} \in E_{\gamma})$, then we can write

v) $J(x) = \sum_{\gamma \in \Gamma} (-1)^{\gamma} x_{\gamma}$.

If $\Gamma = Z$, the additive group of integers, then these definitions agree with the usual definitions of powers of an automorphism, or of an element of an algebra.

§5. **Derivations.** The definition of derivations in a graded algebra given here is somewhat different from the conventional definition of the derivations in the ordinary algebraic systems. In the sequel, when we speak of derivations, we understand that a fixed subgroup $\Gamma_{+} \subset \Gamma$ of index 2 is given.

Now, let E, E' be two Γ-graded algebras over A and let φ be a homomorphism of E into E'.

DEFINITION 1.6. *A φ-derivation D of E into E' means a linear mapping $D: E \to E'$, homogeneous of some given degree $\nu \in \Gamma$, such that for every $x, y \in E$,*

$$D(xy) = D(x)\varphi(y) + \varphi(J^{\nu}x) D(y), \tag{1}$$

where J^{ν} is the power of the main involution defined above.

In the case where $E = E'$ and φ is the identity, D is called simply a "*derivation*". Therefore a derivation D of E is a homogeneous linear mapping of degree ν, such that

$$D(xy) = D(x)y + (J^{\nu}x) D(y) \quad \text{for } x, y \in E \tag{2}$$

If $\Gamma = Z$, the additive group of integers, (2) is written by

(2′) $$D(xy)=D(x)y+(-1)^{h\nu}x\,D(y) \text{ for } x\in E_h,\ y\in E.$$

If the elements of E are all of degree θ (θ : zero element of Γ), then D must be of degree θ, and (2) reduces to

(3) $$D(xy)=D(x)y+x\,D(y),$$

which coincides with the ordinary definition of derivation. Also, when ν belong to Γ_+ (2) reduces to (3), while if ν belong to Γ_- and $x\in E_-^2$, (2) reduces to

(4) $$D(xy)=D(x)y-x\,D(y).$$

A linear mapping satisfying (4) is sometimes called "anti-derivation", but we do not use this terminology in these lectures.

The formula (1) can be written in another form. Denote by L_x the operation of the left multiplication by x: $L_x y=xy$. Then (1) is equivalent to

(5) $$D\circ L_x=L_{D(x)}\circ\varphi+L_{\varphi(J^\nu x)}\circ D.$$

In the case where $E=E'$, and φ is the identity,

(6) $$D\circ L_x=L_{D(x)}+L_{J^\nu x}\circ D.$$

Remark that (5) and (6) do not contain the "parameter" y.

LEMMA 1.5. *For every φ-derivation D, we have $D(1)=0$.*

PROOF. Substituting $x=y=1$ in (1), we get

$$D(1)=D(1\cdot 1)=D(1)\varphi(1)+\varphi(J^\nu 1)\,D(1),$$

and since $J^\nu 1=1$, $\varphi(1)=1$, we obtain $D(1)=D(1)+D(1)$, which proves $D(1)=0$.

Evidently, if D and D' are φ-derivations of the same degree, $D\pm D'$ is again a φ-derivation. Also we have

LEMMA 1.6. *If $\varphi: E\to E'$ and $\varphi': E'\to E''$ are homomorphisms and if D, D' are a φ-derivation of E into E' and a φ'-derivation of E' into E'' respectively, then $\varphi'\circ D$ and $D'\circ\varphi$ are $(\varphi'\circ\varphi)$-derivations of $E\to E''$.*

PROOF. We have only to check the condition (1). By direct calculation we have

$$(\varphi' \circ D)(xy) = \varphi'(D(x))\,\varphi'(\varphi(y)) + \varphi'(\varphi(J^{\nu}x))\varphi'(D(y))$$

and

$$(D' \circ \varphi)(xy) = D'(\varphi(x))\varphi'(\varphi(y)) + \varphi'(\varphi(J^{\nu'}x))D'(\varphi(y)),$$

and since $\varphi' \circ D$ and $D' \circ \varphi$ are of degrees ν and ν' respectively, we have our assertion.

THEOREM 1.4. *Let D be a φ-derivation of E into E', F a homogeneous subalgebra of E, S a set of homogeneous generators of F, and let F' be a homogeneous subalgebra of E'. Then if $D(S) \subset F'$ and $\varphi(S) \subset F'$, we have $D(F) \subset F'$ and $\varphi(F) \subset F'$.*

PROOF. The latter inclusion is evident, because φ is a homomorphism. The former is proved as follows. Let F_1 be the set of elements $x \in F$ such that $D(x) \in F'$. It is evident that F_1 is closed under addition and scalar multiplication. Also if $D(x) \in F'$ and $x = \sum x_\gamma$, then the $D(x_\gamma)$'s are the homogeneous components of $D(x)$ and $D(x_\gamma) \in F'$, so we obtain $x_\gamma \in F_1$. Therefore F_1 is a *homogeneous submodule* of F, so that $x \in F_1$ implies $J^{\nu}x \in F_1$. Now for $x, y \in F_1$, we have

$$D(xy) = D(x)\varphi(y) + \varphi(J^{\nu}x)D(y),$$

and since $D(x)$, $\varphi(y)$, $\varphi(J^{\nu}x)$, $D(y)$ all belong to F', we have $xy \in F_1$, which proves that F_1 is a subalgebra containing S. S being the set of generators of F, we have $F \subset F_1$, which proves $D(F) \subset F'$.

COROLLARY 1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be homogeneous ideals of E and E' respectively, and S be a set of homogeneous generators of $\mathfrak{A}$. If $D(S) \subset \mathfrak{A}'$, $\varphi(S) \subset \mathfrak{A}'$, we have $D(\mathfrak{A}) \subset \mathfrak{A}'$, and $\varphi(\mathfrak{A}) \subset \mathfrak{A}'$.*

PROOF. Again the latter inclusion is evident. The former is proved in a similar manner as before, showing that the set

$$\mathfrak{A}_1 = \{x \mid x \in \mathfrak{A},\ \ D(x) \in \mathfrak{A}'\}$$

is a homogeneous ideal.

COROLLARY 2. *Let F, S be as before. If $D(S) = \{0\}$, then $D(F) = \{0\}$.*[4]

PROOF. In a similar manner as in the proof of Theorem 1.4, we can show that

4) Remark that this assertion holds without any assumption on φ.

$$F_2 = \{x \mid x \in F,\quad D(x)=0\}$$

is a homogeneous subalgebra, which proves $F \subset F_2$.

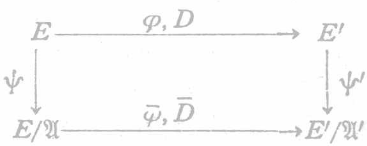
COROLLARY 3. *Let F, S be as before. If two φ-derivations D, D′ coincide with each other on S, then they coincide on F.*

PROOF. From this assumption, D and D' are of the same degree. Then apply Corollary 2 to the derivation $D-D'$.

It follows from this corollary that a derivation D is completely determined if its values on the elements of a set of generators are given.

THEOREM 1.5. *Let E, E′ be Γ-graded algebras, φ a homomorphism $E \to E'$, and D a φ-derivation of $E \to E'$. Also let $\mathfrak{A}$ and $\mathfrak{A}'$ be homogeneous ideals in E and E′ respectively such that $D(\mathfrak{A}) \subset \mathfrak{A}'$, and $\varphi(\mathfrak{A}) \subset \mathfrak{A}'$. Under these assumptions, the induced mapping $\bar{D}$: $E/\mathfrak{A} \to E'/\mathfrak{A}'$ obtained from D is a $\bar{\varphi}$-derivation, where $\bar{\varphi}$ means the induced homomorphism $E/\mathfrak{A} \to E'/\mathfrak{A}'$ obtained from φ.*

If we use the "commutative diagram"[5] the map D and $\bar{\varphi}$ are represented as follows:



where ψ and ψ' are the canonical mappings.

5) In a diagram, let every vertex represent a set, and let each oriented edge represent a mapping. A directed path in a diagram represent a mapping which is the composition of successive mappings assigned to its edges. If, for any two vertices, any two directed paths connecting them give the same mapping, then the diagram is said to be *commutative*. For example in Fig. 1, for the vertices P and Q and the paths as in it, the commutativity means $f_4 \circ f_3 \circ f_2 \circ f_1(x) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1 \circ f_1(x) = f_4 \circ g_6 \circ g_3 \circ g_2 \circ g_1 \circ f_1(x) = \cdots$ for every $x \in P$.