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Linear Algebraic Groups

Second Edition

T. A. Springer

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Preface to the Second Edition

This volume is a completely new version of the book under the same title, which appeared in 1981 as Volume 9 in the series "Progress in Mathematics," and which has been out of print for some time. That book had its origin in notes (taken by Hassan Azad) from a course on the theory of linear algebraic groups, given at the University of Notre Dame in the fall of 1978. The aim of the book was to present the theory of linear algebraic groups over an algebraically closed field, including the basic results on reductive groups. A distinguishing feature was a self-contained treatment of the prerequisites from algebraic geometry and commutative algebra.

The present book has a wider scope. Its aim is to treat the theory of linear algebraic groups over arbitrary fields, which are not necessarily algebraically closed. Again, I have tried to keep the treatment of prerequisites self-contained.

While the material of the first ten chapters covers the contents of the old book, the arrangement is somewhat different and there are additions, such as the basic facts about algebraic varieties and algebraic groups over a ground field, as well as an elementary treatment of Tannaka's theorem in Chapter 2. Errors – mathematical and typographical – have been corrected, without (hopefully) the introduction of new errors. These chapters can serve as a text for an introductory course on linear algebraic groups.

The last seven chapters are new. They deal with algebraic groups over arbitrary fields. Some of the material has not been dealt with before in other texts, such as Rosenlicht's results about solvable groups in Chapter 14, the theorem of Borel of Tits on the conjugacy over the ground field of maximal split torus in an arbitrary linear algebraic group in Chapter 15 and the Tits classification of simple groups over a ground field in Chapter 17.
The prerequisites from algebraic geometry are dealt with in Chapter 11.

I am grateful to many people for comments and assistance: P. Hewitt and Zhe-Xian Wang sent me several years ago lists of corrections of the second printing of the old book, which were useful in preparing the new version. A. Broer, Konstanze Rietsch and W. Soergel communicated lists of comments on the first part of the present book and K. Bongartz, J. C. Jantzen. F. Knop and W. van der Kallen commented on points of detail. The latter also provided me with pictures, and W. Casselman provided Dynkin and Tits diagrams. A de Meijer gave frequent help in coping with the mysteries of computers.

Lastly. I thank Birkhäuser – personified by Ann Kostant– for the help (and patience) with the preparation of this second edition.

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Linear Algebraic Groups

Contents

Chapter 1

Some Algebraic Geometry

This preparatory chapter discusses basic results from algebraic geometry, needed to deal with the elementary theory of algebraic groups. More algebraic geometry will appear as we go along. More delicate results involving ground fields are deferred to Chapter 11.

1.1. The Zariski topology

1.1.1. Let k be an algebraically closed field and put $V = k^n$. The elements of the polynomial algebra $S = k[T_1, \ldots, T_n]$ (abbreviated to $k[T]$) can be viewed as k-valued functions on V. We say that $v \in V$ is a *zero* of $f \in k[T]$ if $f(v) = 0$ and that v is a zero of an ideal I of S if $f(v) = 0$ for all $f \in I$. We denote by $\mathcal{V}(I)$ the set of zeros of the ideal I. If X is any subset of V, let $\mathcal{I}(X) \subset S$ be the ideal of the $f \in S$ with $f(v) = 0$ for all $v \in X$.

Recall that the *radical* or nilradical $\sqrt{I}$ of the ideal I (see [Jac5, p. 392]) is the ideal of the $f \in S$ with $f^n \in I$ for some integer $n \geq 1$. A *radical ideal* is one that coincides with its radical. It is obvious that all $\mathcal{I}(X)$ are radical ideals.

We shall need Hilbert's Nullstellensatz in two equivalent formulations.

1.1.2. Proposition. *(i) If I is a proper ideal in S then $\mathcal{V}(I) \neq \emptyset$;*
(ii) For any ideal I of S we have $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$.

For a proof see for example [La2, Ch. X, §2]. The proposition can also be deduced from the results of 1.9 (see Exercise 1.9.6 (2)).

1.1.3. Zariski topology on V. The function $I \mapsto \mathcal{V}(I)$ on ideals has the following properties:
(a) $\mathcal{V}(\{0\}) = V$, $\mathcal{V}(S) = \emptyset$;
(b) *If $I \subset J$ then $\mathcal{V}(J) \subset \mathcal{V}(I)$;*
(c) $\mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$;
(d) *If $(I_\alpha)_{\alpha \in A}$ is a family of ideals and $I = \sum_{\alpha \in A} I_\alpha$ is their sum, then $\mathcal{V}(I) = \bigcap_{\alpha \in A} \mathcal{V}(I_\alpha)$.*

The proof of these properties is left to the reader (*Hint*: for (c) use that $I.J \subset I \cap J$). It follows from (a), (c) and (d) that there is a topology on V whose closed subsets are the $\mathcal{V}(I)$, I running through the ideals of S. This is the *Zariski topology*. The induced topology on a subset X of V is the Zariski topology of V. A closed set in V is called an *algebraic set*.

1.1.4. Exercises. (1) Let $V = k$. The proper algebraic sets are the finite ones.

(2) The Zariski closure of $X \subset V$ is $\mathcal{V}(\mathcal{I}(X))$.
(3) The map $\mathcal{I}$ defines an order reversing bijection of the family of Zariski closed subsets of V onto the family of radical ideals of S. Its inverse is $\mathcal{V}$.
(4) The Euclidean topology on $\mathbf{C}^n$ is finer than the Zariski topology.

1.1.5. Proposition. *Let $X \subset V$ be an algebraic set.*
(i) The Zariski topology of X is T_1, i.e., points are closed;
(ii) Any family of closed subsets of X contains a minimal one;
(iii) If $X_1 \supset X_2 \supset \ldots$ is a descending sequence of closed subsets of X, there is an h such that $X_i = X_h$ for $i \geq h$;
(iv) Any open covering of X has a finite subcovering.

If $x = (x_1, \ldots, x_n) \in X$ then x is the unique zero of the ideal of S generated by $T_1 - x_1, \ldots, T_n - x_n$. This implies (i). (ii) and (iii) follow from the fact that S is a Noetherian ring [La2, Ch. VI, §1], using 1.1.4 (3).

To establish assertion (iv) we formulate it in terms of closed sets. We then have to show: if $(I_\alpha)_{\alpha \in A}$ is a family of ideals such that $\bigcap_{\alpha \in A} \mathcal{V}(I_\alpha) = \emptyset$, there is a finite subset B of A such that $\bigcap_{\alpha \in B} \mathcal{V}(I_\alpha) = \emptyset$. Using properties (a), (d) of 1.1.3 and 1.1.4 (3) we see that $\sum_{\alpha \in A} I_\alpha = S$. There are finitely many of the I_α, say $I_1, \ldots, I_h$, such that 1 lies in their sum. It follows that $I_1 + \ldots + I_h = S$, which implies that $\bigcap_{i=1}^h \mathcal{V}(I_i) = \emptyset$. □

A topological space X with the property (ii) is called *noetherian*. Notice that (ii) and (iii) are equivalent properties (compare the corresponding properties in noetherian rings, cf. [La2, p. 142]. X is *quasi-compact* if it has the property of (iv).

1.1.6. Exercise. A closed subset of a noetherian space is noetherian for the induced topology.

1.2. Irreducibility of topological spaces

1.2.1. A topological space X (assumed to be non-empty) is *reducible* if it is the union of two proper closed subsets. Otherwise X is *irreducible*. A subset $A \subset X$ is irreducible if it is irreducible for the induced topology. Notice that X is irreducible if and only if any two non-empty open subsets of X have a non-empty intersection.

1.2.2. Exercise. An irreducible Hausdorff space is reduced to a point.

1.2.3. Lemma. *Let X be a topological space.*
(i) $A \subset X$ is irreducible if and only if its closure $\bar{A}$ is irreducible;
(ii) Let $f : X \to Y$ be a continuous map to a topological space Y. If X is irreducible then so is the image fX.

Let A be irreducible. If $\bar{A}$ is the union of two closed subsets A_1 and A_2 then A is