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Seán Dineen

Complex Analysis on Infinite Dimensional Spaces

无限维空间上的复分析

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To

the memory

of

Paul Newbrough

Preface

Infinite dimensional holomorphy is the study of holomorphic or analytic functions over complex topological vector spaces. The terms in this description are easily stated and explained and allow the subject to project itself initially, and innocently, as a compact theory with well defined boundaries. However, a comprehensive study would include delving into, and interacting with, not only the obvious topics of topology, several complex variables theory and functional analysis but also, differential geometry, Jordan algebras, Lie groups, operator theory, logic, differential equations and fixed point theory. This diversity leads to a dynamic synthesis of ideas and to an appreciation of a remarkable feature of mathematics - its unity. Unity requires synthesis while synthesis leads to unity.

It is necessary to stand back every so often, to take an overall look at one's subject and ask "How has it developed over the last ten, twenty, fifty years? Where is it going? What am I doing?" I was asking these questions during the spring of 1993 as I prepared a short course to be given at Universidade Federal do Rio de Janeiro during the following July. The abundance of suitable material made the selection of topics difficult. For some time I hesitated between two very different aspects of infinite dimensional holomorphy, the geometric-algebraic theory associated with bounded symmetric domains and Jordan triple systems and the topological theory which forms the subject of the present book. I did not intend to write a book and, so, the choice of topic did not appear to have long term consequences. I had written a book[1] on locally convex structures on spaces of holomorphic functions some fifteen years previously and did not believe that the area has changed sufficiently to warrant another book. I took the apparently easy option of surveying recent developments, but while preparing my notes I was pleasantly surprised and, by the end of 1993, I knew that this book should be written and that it would take some time. It took almost six years and would never have been written without my biannual visits to Rio de Janeiro, kindly arranged at different times by Jorge Alberto Barroso, Luiza Moraes and Roberto Soraggi, where I had the opportunity to lecture on most of the material.

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[1] *Complex Analysis on Locally Convex Spaces*, North Holland Math. Studies, 57, 1981.

The central theme in this book is the relationship on $H(U)$, the holomorphic functions on an open subset U of a locally convex space E, between the three topologies, τ_0 (the compact open topology), τ_ω (the ported or Nachbin topology) and τ_δ (the topology generated by the countable open covers). The portrayal of topologies as structures whose function is to define modes of convergence and continuity tends to obscure other roles they play. A topology is also a *selection* process which replaces the study of random subsets by *identifying* those likely to display interesting features. Different topologies give rise to different selections and reveal different features of the ambient space. The two extreme topologies considered here are τ_0 and τ_δ. The compact open topology has a function theoretic pedigree and is more likely to be useful, at least initially, with problems having their roots in several complex variables theory. It is, however, based on compact subsets of the underlying (domain) space and, in general, compact subsets of an infinite dimensional space are *small* and, consequently, less influential. On the other hand, open subsets are *large*, almost too large, and the τ_δ topology, based on such sets, can be a crude selection process when dealing with holomorphic functions. This mixture of ideas and concepts from different areas is, however, full of surprises and we find in Chapter 5 the following redeeming function theoretic property of the τ_δ topology:

> *if Ω_1 and Ω_2 are domains spread over a Fréchet space and Ω_2 is an analytic extension of Ω_1 then the bijective mapping, defined by analytic continuation, is always a τ_δ isomorphism*

(but may not be a τ_0 isomorphism).

Complications also arise on the linear side from the unavoidable presence of nuclearity in any reasonable topology on $H(U)$. This point is clearly and easily illustrated when E is a Banach space. In this case $(H(E), \tau_\delta)$ contains Fréchet nuclear spaces and Banach spaces as *complemented* subspaces – an unusual combination – intuitively avoided by the functional analyst. On the other hand the τ_0 topology on $H(E)$ has the useful property that its closed bounded sets are compact but has the drawback that it induces on E' (the continuous linear functions on E) a rather weak topology.

The ideal situation occurs when $\tau_0 = \tau_\delta$ and we have a topology which is acceptable in function theory and functional analysis. Examples are given in Chapter 4. These are important and include both $\mathcal{DFM}$ spaces and Fréchet nuclear spaces with (DN) but exclude *all* infinite dimensional Banach spaces. Between τ_0 and τ_δ we encounter the *intermediate* τ_ω topology, defined using neighbourhood systems of compact sets. It is sufficiently close to τ_0 to inherit some of its good behaviour and, yet, sufficiently removed, from τ_0 towards τ_δ, to potentially share properties with τ_δ. We may regard τ_ω as a compromise between the conflicting suggestions of several complex variables theory and linear functional analysis.

In spite of these conflicts and complications, or perhaps because of them, the results are surprisingly positive. Investigations, over many years, have shown that certain natural splittings, approximations and size restrictions need to be in place to obtain positive solutions to the topological equations $\tau_0 = \tau_\delta$ and $\tau_\omega = \tau_\delta$. Fortunately, these conditions are present in what are generally regarded as the most interesting locally convex spaces. The main results for these topologies are presented in the two most technically demanding chapters, four and five. The results for balanced domains – where the key to success is pointwise convergence of the Taylor series expansion at *all* points of the domain – are fairly complete and discussed in Chapter 4. The Taylor series expansion facilitates the employment of techniques from functional analysis but as we move from this setting there is a steady drift, especially in the final two chapters, towards non-linear methods and involvement with $H(U)$ as an algebra. The theory for arbitrary open sets, presented in Chapter 5, contains significant positive results and many open problems. Even when these topologies do not coincide they can be used in tandem to uncover what might otherwise have remained hidden, e.g. τ_δ, τ_ω and τ_0 combine to present $H(E)$ as a dual space.

Somehow, and surprisingly in view of its initial modest aims, the topological problems we consider capture the tension between the *finite* dimensional *holomorphic* theory and the *infinite* dimensional *linear* theory and, acting as a catalyst, fuse from them a topic with its own internal logic and intrinsic unity. Thus, in examining the basic definitions and considering fundamental topological questions we encounter in a natural and essential way such diverse topics as the bounded approximation property, finite dimensional decompositions, the Dunford–Pettis property, the Radon–Nikodým Property, the principle of local reflexivity, ultrapowers, (BB)-property, (DN)-property, the density condition, Arens regularity, hypocontinuity, spreading models, determining sets, the Levi problem, and meet new intrinsic concepts such as polarization constants, bounding sets, uniform factorization, compact non-polar sets, $\mathcal{S}$-absolute decompositions, Taylor series completeness, entire functions of bounded type, etc. The answers that resulted from these topological questions (for instance in Section 5.2 we required almost the complete solution to the Levi problem in order to obtain $\tau_0 = \tau_\omega$ on open subsets of Fréchet–Schwartz spaces) and the presence of positive results suggest that infinite dimensional holomorphy will not be hindered by topological obstructions and, indeed, will be positively enriched when such considerations enter the picture.

This book is divided into six chapters, each devoted to a single theme. Chapters 1 and 3 introduce and cover the basic properties of polynomials and holomorphic functions over locally convex spaces respectively. With these two chapters as reference, the other four become almost independent self-contained units which complement one another and taken together add to the overall structure of the subject and book. The first two chapters are a self-contained study of polynomials (today an essentially independent field of investigation within linear functional analysis). This area has seen rapid development over the last ten years and the wide choice of material available obliged us to omit some interesting topics. In Chapter 1 we develop the basic theory, using tensor products, and discuss geometric properties of polynomials on Banach spaces. Chapter 2 is devoted to duality theory for different spaces of polynomials. Chapter 3 discusses Taylor and monomial expansions of Gâteaux and Fréchet holomorphic functions while Chapters 4 and 5 concentrate on relationships between the topologies τ_0, τ_ω and τ_δ. Chapter 6 examines the interplay between various concepts that were uncovered, in earlier chapters, as being intrinsic to infinite dimensional holomorphy.

Each chapter contains text, a set of exercises and a final section of notes. The exercises, notes, and appendix (which contains remarks on selected exercises) allowed us to insert material which, in the main text, would have interrupted the flow of essential material and led to the inclusion of excessive detail. In the notes and appendix we provide information on the history of the subject and references for the material presented. We have tried to be as careful as possible in this regard and take responsibility for the inevitable errors. Accurate and comprehensive records of this kind are not a luxury but essential background information in appreciating and understanding a subject and its evolution. Authors, who do not take this aspect of their work seriously, devalue their chosen subject and, ultimately, their own contribution.

We assume the reader has a basic knowledge of one complex variable theory and some experience with Banach space theory. The reader familiar with several complex variables and locally convex spaces will undoubtedly find the subject less difficult but we include definitions and results from these areas as required. We have tried to maintain a delicate balance between our desire to write a self-contained introduction for the non-expert and to provide a comprehensive summary for the expert.

It is a pleasure, and a relief, to arrive at the stage where I can thank those who helped me in this project. The many mathematicians who organized conferences and published proceedings over the years in this area performed a much appreciated service and facilitated my task enormously. The September 1994 Dublin conference on "*Polynomials and Holomorphic Functions over Infinite Dimensional Spaces*" occurred at a crucial time and the excellent survey lectures and the set of problems circulated at that conference played a key role in convincing me to continue writing this book. The participants at the weekly University College Dublin–Trinity College Dublin Analysis Seminar displayed remarkable patience, while I experimented with my presentation, and contributed with their honest and helpful advice. The analysis group at Universidade Federal do Rio de Janeiro, Roberto Soraggi, Luiza Moraes and Jorge Alberto Barroso, deserve to be mentioned in the introduction to each chapter for the wonderful hospitality and support they provided over the full period during which this book was written. The intensive courses in Complex Analysis, sponsored by the Erasmus Programme, organized initially by the Galois Network and Frank de Clerck (Ghent) and continued by Jaime

Carvalho de Silva (Coimbra) provided me with the opportunity to give short courses on some of the material in the delightful city of Coimbra. Financial support for some of these visits was provided by UFRJ (Universidade Federal do Rio de Janeiro), FAPERJ (Fundacaõ de Amparo a Pesquisa do Estado do Rio de Janeiro), CNPq (Conselho Nacional de Desenvolvimento Cientifica e Tecnologico), the Erasmus Programme of the European Union and the Faculty of Arts at University College Dublin. I am particularly grateful to the Modular Degree Programme at University College Dublin. The income from the night courses I gave in this programme was the only support I had to transform my handwritten notes into printed form.

Many individuals helped, with their technical advice and mathematical expertise, in the preparation of different chapters of this book but Jose Ansemil, Chris Boyd, Michael Mackey, Pilar Rueda and Thomas Unger gave unselfishly of their time with *all* chapters and I would like to single them out for special thanks. Their influence has been enormous and so pervasive that it is now impossible to detail. Raymundo Alencar, Richard Aron, Fernando Blasco, Yung Sung Choi, Veronica Dimant, Klaus Floret, José Isidro, Manolo Maestre, Pauline Mellon, Jorge Mujica, Yannis Sarantopoulos, Ray Ryan, Richard Timoney and Nacho Zalduendo provided specialized advice and encouragement when it mattered.

Finally, a special paragraph for Dana, soon to become Dr. Nicolau, who must have felt, at some points, that my revisions were not converging and that I was taking the "infinite" in the title too literally. Despite these reservations she did an excellent job in preparing this book for publication in the midst of a very busy period in her own studies. Thank you, Dana.

Susan Hezlet, recently of Springer-Verlag but now with the London Mathematical Society, is the type of editor that every author should have - helpful, realistic and encouraging. Her successor at Springer-Verlag, David Ireland, has been helpful and understanding during the final phase of this project.

Finally, a special word of thanks to the Dean of the Faculty of Arts, Professor Fergus D'Arcy, whose personal support and appreciation of scholarship and creativity does make a difference.

This book took a long time to write - much longer than planned. By reading it you will be thanking all those who helped me.

University College Dublin,
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Chapter 1. Polynomials

Multilinear mappings, tensor products, restrictions to finite dimensional spaces and differential calculus may all be used to define polynomials over infinite dimensional spaces. All of these are useful, none should be neglected and, indeed, the different possible approaches and interpretations add to the richness of the subject. We adopt an integrated approach to the development of polynomials using multilinear mappings and tensor products. The philosophy of tensor products is easily stated: to exchange polynomial functions on a given space with *simpler* (linear) *functions* on a (possibly) more *complicated space*. As a typical example (see Section 1.2) we shall see that the space of continuous n-homogeneous polynomials on the locally convex space E can be realized as the space of continuous linear functions on the n-fold symmetric projective tensor product of E, $\widehat{\bigotimes}_{n,s,\pi} E$.

In Section 1.1 we begin by studying, at the algebraic level, the relationship between symmetric n-linear forms, symmetric n-tensors and n-homogeneous polynomials. Afterwards we refine this relationship to the continuous level by considering continuous polynomials between locally convex spaces and a further refinement arises, in Section 1.2, when we discuss topologies on spaces of polynomials. At the algebraic level we discuss linearization, duality and the polarization formula while, at the continuous level, we also meet factorization. This perspective leads to the recasting of certain results, e.g. (1.16) is just a translation of the polarization formula into the language of symmetric tensors, and to an examination, in Section 1.2, of the different topologies as uniform convergence over sets of tensors. Section 1.3, which begins as a more detailed analysis of the polarization inequality, ends up as a study of the interplay between polynomials and the *isometric* (or *geometric*) properties of Banach spaces - Chapter 2 is mainly devoted to the relationship between polynomials and *isomorphic* (or *linear topological*) properties of Banach spaces. As we proceed we require various concepts and results from locally convex space theory (Section 1.2) and Banach space theory (Section 1.3). These are introduced, without proof, as required, in a format deemed suitable from the perspective of infinite dimensional holomorphy.

1.1 Continuous Polynomials

$\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ denote respectively the complex numbers, the real numbers, the natural numbers and the integers. If A and B are sets, and $n, m \in \mathbb{N}$ then A^nB^m will denote the Cartesian product of n copies of A and m copies of B and x^ny^m will denote the element $(\underbrace{x,\ldots,x}_{n \text{ times}},\underbrace{y,\ldots,y}_{m \text{ times}})$. E and F will denote vector spaces over $\mathbb{C}$.

For $n \in \mathbb{N}$ we let $\mathcal{L}_a(^nE;F)$ denote the *space of n-linear mappings* from E into F. The subscript a denotes *algebraic* since we do not assume any continuity properties. Hence, if $L \in \mathcal{L}_a(^nE;F)$, then L is an F-valued function defined on E^n which is linear in each variable when the remaining $(n-1)$ variables are fixed. Clearly, $\mathcal{L}_a(^nE;F)$ is a vector space over $\mathbb{C}$. 1-linear mappings are just linear mappings and in this case we use the notation $\mathcal{L}_a(E;F)$. 2-linear mappings are also called bilinear mappings and certain authors use the notation $\mathcal{B}_a(E;F)$ in place of $\mathcal{L}_a(^2E;F)$. The notation $\mathcal{B}(E)$ is also used in the literature to denote the set of all bounded linear mappings from the Banach space E into itself. When $F = \mathbb{C}$ we write $\mathcal{L}_a(^nE)$ in place of $\mathcal{L}_a(^nE;\mathbb{C})$ and E^* in place of $\mathcal{L}_a(E;\mathbb{C})$. E^* is called the *algebraic dual of E*. When $n = 0$ we define $\mathcal{L}_a(^0E;F)$ to be the set of constant mappings from E into F and this space can be identified with F in a natural fashion. If f is an F-valued function, F is a vector space over $\mathbb{C}$, α is a semi-norm on F and A is contained in the domain of f we let $\|f\|_{\alpha,A} = \sup_{x\in A} \alpha(f(x))$. If α is clearly understood from the context we just write $\|f\|_A$.

The following useful algebraic identities are easily established by induction.

Proposition 1.1 *If E, F and G are vector spaces over $\mathbb{C}$ and $m, n \in \mathbb{N}$ then the mappings I_m and J_m, defined as follows, are linear isomorphisms:*

$$I_m: \mathcal{L}_a(^{m+n}E;F) \longrightarrow \mathcal{L}_a(^mE;\mathcal{L}_a(^nE;F))$$

$$[I_mA(x)](y) := A(x,y),$$

where $A \in \mathcal{L}_a(^{m+n}E;F)$, $x \in E^m$ and $y \in E^n$,

$$J_m: \mathcal{L}_a(^mE;\mathcal{L}_a(^nF;G)) \longrightarrow \mathcal{L}_a(^nF;\mathcal{L}_a(^mE;G))$$

$$[J_mA(y)](x) := [A(x)](y),$$

where $A \in \mathcal{L}_a(^mE;\mathcal{L}_a(^nF;G))$, $x \in E^m$, $y \in F^n$.

A particular case, often used in the linear theory, is the isomorphism I_1 which gives, when $n = 1$ and $F = \mathbb{C}$,

$$\mathcal{L}_a(^2E) \approx \mathcal{L}_a(E;E^*).$$