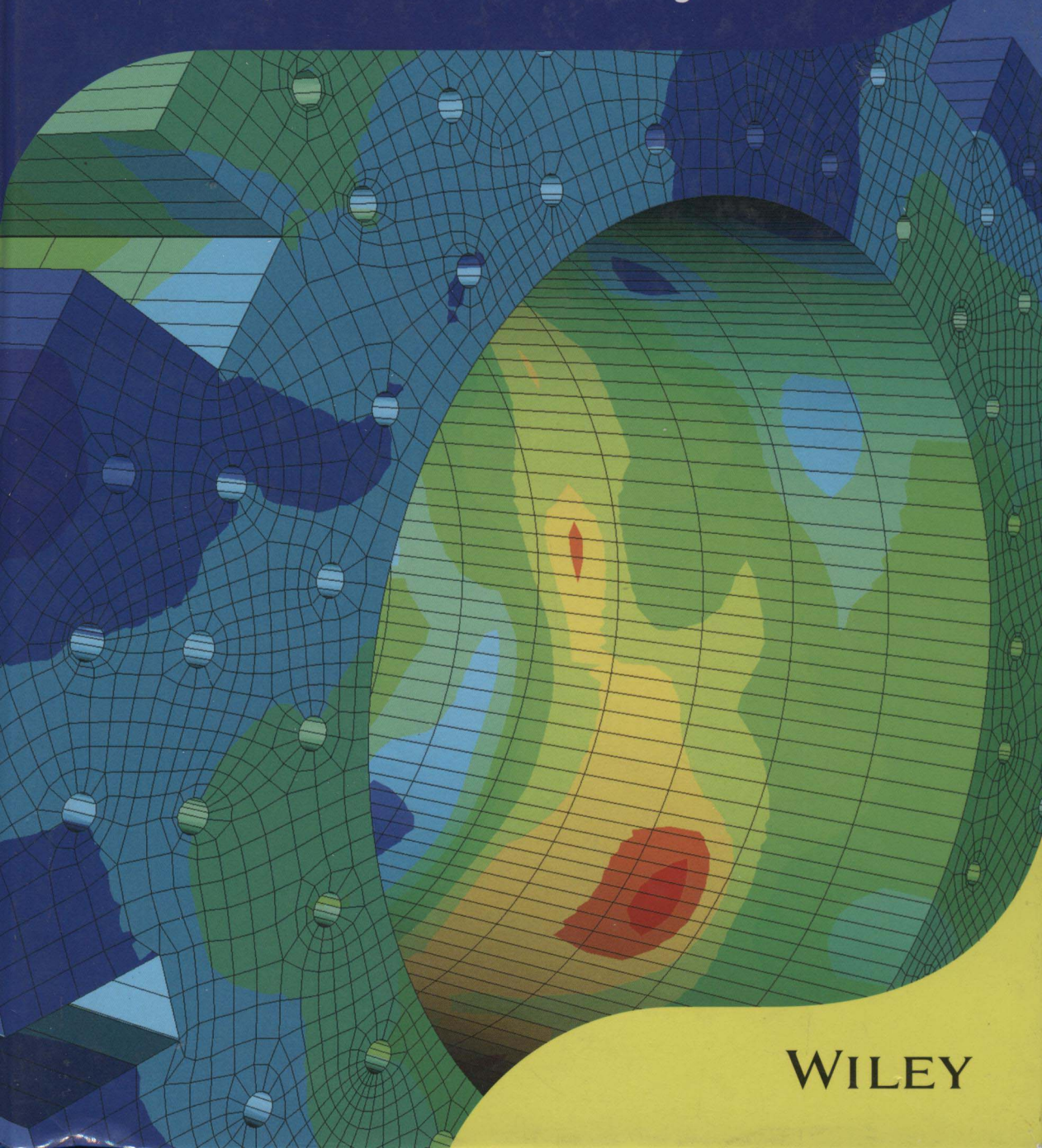


Fifth Edition

# Programming the Finite Element Method

I. M. Smith, D. V. Griffiths and L. Margetts



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# PROGRAMMING THE FINITE ELEMENT METHOD

*Fifth Edition*

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# **PROGRAMMING THE FINITE ELEMENT METHOD**



# Preface to Fifth Edition

This edition maintains the successful theme of previous editions, namely a modular programming style which leads to concise, easy to read computer programs for the solution of a wide range of problems in engineering and science governed by partial differential equations.

The programming style has remained essentially the same despite huge advances in computer hardware. Readers will include beginners, making acquaintance with the finite element method for the first time, and specialists solving very large problems using the latest generation of parallel supercomputers.

In this edition special attention is paid to interfacing with other open access software, for example ParaView for results visualisation, ABAQUS user subroutines for a range of material constitutive models, ARPACK for large eigenvalue analyses, and METIS for mesh partitioning.

Chapter 1 has been extensively rewritten to take account of rapid developments in computer hardware, for example the availability of GPUs and cloud computing environments. In Chapters 2 to 11 numerous additions have been made to enhance analytical options, for example new return algorithms for elastoplastic analyses, more general boundary condition specification and a complex response option for dynamic analyses.

Chapter 12 has been updated to illustrate the rapidly advancing possibilities for finite element analyses in parallel computing environments. In the fourth edition the maximum number of parallel ‘processes’ used was 64 whereas in this edition the number has increased to 64,000. The use of GPUs to accelerate computations is illustrated.





# Acknowledgements

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There were many contributions to Chapter 12. Llion Evans, Paul Mummery, Philip Manning, Graham Hall and Dimitris Christias (all University of Manchester) provided scientific case studies. Florent Lebeau and Francois Bodin (CAPS Entreprise) evaluated the use of GPUs and Philippe Young (Simpleware Ltd) provided support in image-based modelling.

Benchmarking of the programs in Chapter 12 was carried out using supercomputers hosted by the UK National High Performance Computing Service “HECToR” (e107, e254) and the UK Regional Service “N8 HPC” (EP/K000225/1). The EU FP7 project “Venus-C” and Barcelona Supercomputing Center (Spain) provided access, resources and training to use Microsoft Azure.

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