天元基金影印数学丛书

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Functional Analysis for Probability and Stochastic Processes

概率论与随机过程中的泛函分析（影印版）

Adam Bobrowski

高等教育出版社
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序 言

为了更好地借鉴国外数学教育与研究的成功经验，促进我国数学教育与研究事业的发展，提高高等学校数学教育教学质量，本着“为我国热爱数学的青年创造一个较好的学习数学的环境”这一宗旨，天元基金赞助出版“天元基金影印数学丛书”。

该丛书主要包含国外反映近代数学发展的纯数学与应用数学方面的优秀书籍，天元基金邀请国内各个方向的知名数学家参与选题的工作，经专家遴选、推荐，由高等教育出版社影印出版。为了提高我国数学研究生教学的水平，暂把选书的目标确定在研究生教材上。当然，有的书也可作为高年级本科生教材或参考书，有的书则介于研究生教材与专著之间。

欢迎各方专家、读者对本丛书的选题、印刷、销售等工作提出批评和建议。

天元基金领导小组

2007 年 1 月

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Preface

This book is an expanded version of lecture notes for the graduate course "An Introduction to Methods of Functional Analysis in Probability and Stochastic Processes" that I gave for students of the University of Houston, Rice University, and a few friends of mine in Fall, 2000 and Spring, 2001. It was quite an experience to teach this course, for its attendees consisted of, on the one hand, a group of students with a good background in functional analysis having limited knowledge of probability and, on the other hand, a group of statisticians without a functional analysis background. Therefore, in presenting the required notions from functional analysis, I had to be complete enough for the latter group while concise enough so that the former would not drop the course from boredom. Similarly, for the probability theory, I needed to start almost from scratch for the former group while presenting the material in a light that would be interesting for the latter group. This was fun. Incidentally, the students adjusted to this challenging situation much better than I.

In preparing these notes for publication, I made an effort to make the presentation self-contained and accessible to a wide circle of readers. I have added a number of exercises and disposed of some. I have also expanded some sections that I did not have time to cover in detail during the course. I believe the book in this form should serve first year graduate, or some advanced undergraduate students, well. It may be used for a two-semester course, or even a one-semester course if some background is taken for granted. It must be made clear, however, that this book is not a textbook in probability. Neither may it be viewed as a textbook in functional analysis. There are simply too many important subjects in these vast theories that are not mentioned here. Instead, the book is intended for those who would like to see some aspects of probability from the perspective of functional analysis. It may also serve as a (slightly long) introduction to such excellent and comprehensive expositions of probability and stochastic processes as Stroock's, Revuz's and Yor's, Kallenberg's or Feller's.

It should also be said that, despite its substantial probabilistic content, the book is not structured around typical probabilistic problems and methods. On the contrary, the structure is determined by notions that are functional analytic in origin. As it may be seen from the very chapters' titles, while the body is probabilistic, the skeleton is functional analytic.

Most of the material presented in this book is fairly standard, and the book is meant to be a textbook and not a research monograph. Therefore, I made little or no effort to trace the source from which I had learned a particular theorem or argument. I want to stress, however, that I have learned this material from other mathematicians, great and small, in particular by reading their books. The bibliography gives the list of these books, and I hope it is complete. See also the bibliographical notes to each chapter. Some examples, however, especially towards the end of the monograph, fit more into the category of "research".

A word concerning prerequisites: to follow the arguments presented in the book the reader should have a good knowledge of measure theory and some experience in solving ordinary differential equations. Some knowledge of abstract algebra and topology would not hurt either. I sketch the needed material in the introductory Chapter 1. I do not think, though, that the reader should start by reading through this chapter. The experience of going through prerequisites before diving into the book may prove to be like the one of paying a large bill for a meal before even tasting it. Rather, I would suggest browsing through Chapter 1 to become acquainted with basic notation and some important examples, then jumping directly to Chapter 2 and referring back to Chapter 1 when needed.

I would like to thank Dr. M. Papadakis, Dr. C. A. Shaw, A. Renwick and F. J. Foss (both PhDs soon) for their undivided attention during the course, efforts to understand Polish-English, patience in endless discussions about the twentieth century history of mathematics, and valuable impact on the course, including how-to-solve-it-easier ideas. Furthermore, I would like to express my gratitude to the Department of Mathematics at UH for allowing me to teach this course. The final chapters of this book were written while I held a special one-year position at the Institute of Mathematics of the Polish Academy of Sciences, Warsaw, Poland.

A final note: if the reader dislikes this book, he/she should blame F. J. Foss who nearly pushed me to teach this course. If the reader likes it, her/his warmest thanks should be sent to me at both addresses: bobrowscy@op.pl and a.bobrowski@pollub.pl. Seriously, I would like to thank Fritz Foss for his encouragement, for valuable feedback and for editing parts of this book. All the remaining errors are protected by my copyright.

1
Preliminaries, notations and conventions

Finite measures and various classes of functions, including random variables, are examples of elements of natural Banach spaces and these spaces are central objects of functional analysis. Before studying Banach spaces in Chapter 2, we need to introduce/recall here the basic topological, measure-theoretic and probabilistic notions, and examples that will be used throughout the book. Seen from a different perspective, Chapter 1 is a big "tool-box" for the material to be covered later.

1.1 Elements of topology

1.1.1 *Basics of topology* We assume that the reader is familiar with basic notions of topology. To set notation and refresh our memory, let us recall that a pair $(S,\mathcal{U})$ where S is a set and $\mathcal{U}$ is a collection of subsets of S is said to be a **topological space** if the empty set and S belong to $\mathcal{U}$, and unions and finite intersections of elements of $\mathcal{U}$ belong to $\mathcal{U}$. The family $\mathcal{U}$ is then said to be the **topology** in S, and its members are called **open sets**. Their complements are said to be **closed**. Sometimes, when $\mathcal{U}$ is clear from the context, we say that the set S itself is a topological space. Note that all statements concerning open sets may be translated into statements concerning closed sets. For example, we may equivalently define a topological space to be a pair $(S,\mathcal{C})$ where $\mathcal{C}$ is a collection of sets such that the empty set and S belong to $\mathcal{C}$, and intersections and finite unions of elements of $\mathcal{C}$ belong to $\mathcal{C}$.

An open set containing a point $s \in S$ is said to be a **neighborhood** of s. A topological space $(S,\mathcal{U})$ is said to be **Hausdorff** if for all $p_1, p_2 \in S$, there exists $A_1, A_2 \in \mathcal{U}$ such that $p_i \in A_i, i = 1,2$ and $A_1 \cap A_2 = \emptyset$. Unless otherwise stated, we assume that all topological spaces considered in this book are Hausdorff.

The **closure**, $cl(A)$, of a set $A \subset S$ is defined to be the smallest closed set that contains A. In other words, $cl(A)$ is the intersection of all closed sets that contain A. In particular, $A \subset cl(A)$. A is said to be **dense** in S iff $cl(A) = S$.

A family $\mathcal{V}$ is said to be a **base** of topology $\mathcal{U}$ if every element of $\mathcal{U}$ is a union of elements of $\mathcal{V}$. A family $\mathcal{V}$ is said to be a **subbase** of $\mathcal{U}$ if the family of finite intersections of elements of $\mathcal{V}$ is a base of $\mathcal{U}$.

If $(S,\mathcal{U})$ and $(S',\mathcal{U}')$ are two topological spaces, then a map $f : S \to S'$ is said to be **continuous** if for any open set A' in $\mathcal{U}'$ its inverse image $f^{-1}(A')$ is open in S.

Let S be a set and let $(S',\mathcal{U}')$ be a topological space, and let $\{f_t, t \in \mathbb{T}\}$ be a family of maps from S to S' (here $\mathbb{T}$ is an abstract indexing set). Note that we may introduce a topology in S such that all maps f_t are continuous, a trivial example being the topology consisting of all subsets of S. Moreover, an elementary argument shows that intersections of finite or infinite numbers of topologies in S is a topology. Thus, there exists the smallest topology (in the sense of inclusion) under which the f_t are continuous. This topology is said to be **generated** by the family $\{f_t, t \in \mathbb{T}\}$.

1.1.2 **Exercise** Prove that the family $\mathcal{V}$ composed of sets of the form $f_t^{-1}(A'), t \in \mathbb{T}, A' \in \mathcal{U}'$ is a subbase of the topology generated by $f_t, t \in \mathbb{T}$.

1.1.3 *Compact sets* A subset K of a topological space $(S,\mathcal{U})$ is said to be **compact** if every open cover of K contains a finite subcover. This means that if $\mathcal{V}$ is a collection of open sets such that $K \subset \bigcup_{B\in\mathcal{V}} B$, then there exists a finite collection of sets $B_1, \ldots, B_n \in \mathcal{V}$ such that $K \subset \bigcup_{1=1}^n B_i$. If S is compact itself, we say that the space $(S,\mathcal{U})$ is compact (the reader may have noticed that this notion depends as much on S as it does on $\mathcal{U}$). Equivalently, S is compact if, for any family $C_t, t \in \mathbb{T}$ of closed subsets of S such that $\bigcap_{t\in\mathbb{T}} C_t = \emptyset$, there exists a finite collection $C_{t_1}, \ldots, C_{t_n}$ of its members such that $\bigcap_{i=1}^n C_{t_i} = \emptyset$. A set K is said to be **relatively compact** iff its closure is compact. A topological space $(S,\mathcal{U})$ is said to be **locally compact** if for every point $p \in S$ there exist an open set A and a compact set K, such that $s \in A \subset K$. The **Bolzano–Weierstrass Theorem** says that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded. In particular, $\mathbb{R}^n$ is locally compact.

1.1.4 *Metric spaces* Let $\mathbb{X}$ be an abstract space. A map $d : \mathbb{X}\times\mathbb{X} \to \mathbb{R}^+$ is said to be a **metric** iff for all $x, y, z \in \mathbb{X}$

(a) $d(x, y) = d(y, x)$,
(b) $d(x, y) \le d(x, z) + d(z, y)$,
(c) $d(x, y) = 0$ iff $x = y$.

A sequence x_n of elements of $\mathbb{X}$ is said to **converge** to $x \in \mathbb{X}$ if $\lim_{n\to\infty} d(x_n, x) = 0$. We call x the **limit** of the sequence $(x_n)_{n\ge1}$ and write $\lim_{n\to\infty} x_n = x$. A sequence is said to be **convergent** if it converges to some x. Otherwise it is said to be **divergent**.

An **open ball** $B(x, r)$ with radius r and center x is defined as the set of all $y \in \mathbb{X}$ such that $d(x, y) < r$. A closed ball with radius r and center x is defined similarly as the set of y such $d(x, y) \le r$. A natural way to make a metric space into a topological space is to take all open balls as the base of the topology in $\mathbb{X}$. It turns out that under this definition a subset A of a metric space is closed iff it contains the limits of sequences with elements in A. Moreover, A is compact iff every sequence of its elements contains a converging subsequence and its limit belongs to the set A. (If S is a topological space, this last condition is necessary but not sufficient for A to be compact.)

A function $f : \mathbb{X} \to \mathbb{Y}$ that maps a metric space $\mathbb{X}$ into a normed space $\mathbb{Y}$ is continuous at $x \in \mathbb{X}$ if for any sequence x_n converging to x, $\lim_{n\to\infty} f(x_n)$ exists and equals $f(x)$ (x_n converges in $\mathbb{X}$, $f(x_n)$ converges in $\mathbb{Y}$). f is called continuous if it is continuous at every $x \in \mathbb{X}$ (this definition agrees with the definition of continuity given in 1.1.1).

1.2 Measure theory

1.2.1 *Measure spaces and measurable functions* Although we assume that the reader is familiar with the rudiments of measure theory as presented, for example, in [103], let us recall the basic notions. A family $\mathcal{F}$ of subsets of an abstract set Ω is said to be a **σ-algebra** if it contains Ω and complements and countable unions of its elements. The pair $(\Omega, \mathcal{F})$ is then said to be a **measurable space**. A family $\mathcal{F}$ is said to be an **algebra** or a **field** if it contains Ω, complements and finite unions of its elements.

A function μ that maps a family $\mathcal{F}$ of subsets of Ω into $\mathbb{R}^+$ such that

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.1}$$

for all pairwise-disjoint elements $A_n, n \in \mathbb{N}$ of $\mathcal{F}$ such that the union $\bigcup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{F}$ is called a **measure**. In most cases $\mathcal{F}$ is a σ-algebra but there are important situations where it is not, see e.g. 1.2.8 below. If $\mathcal{F}$ is a σ-algebra, the triple $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**.

Property (1.1) is termed **countable additivity**. If $\mathcal{F}$ is an algebra and $\mu(S) < \infty$, (1.1) is equivalent to

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad \text{whenever } A_n \in \mathcal{F}, A_n \supset A_{n+1}, \bigcap_{n=1}^{\infty} A_n = \emptyset. \quad (1.2)$$

The reader should prove it.

The smallest σ-algebra containing a given class $\mathcal{F}$ of subsets of a set is denoted $\sigma(\mathcal{F})$. If Ω is a topological space, then $\mathcal{B}(\Omega)$ denotes the smallest σ-algebra containing open sets, called the **Borel σ-algebra**. A measure μ on a measurable space $(\Omega, \mathcal{F})$ is said to be **finite** (or **bounded**) if $\mu(\Omega) < \infty$. It is said to be **σ-finite** if there exist measurable subsets Ω_n, $n \in \mathbb{N}$, of Ω such that $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$.

A measure space $(\Omega, \mathcal{F}, \mu)$ is said to be **complete** if for any set $A \subset \Omega$ and any measurable B conditions $A \subset B$ and $\mu(B) = 0$ imply that A is measurable (and $\mu(A) = 0$, too). When Ω and $\mathcal{F}$ are clear from the context, we often say that the measure μ itself is complete. In Exercise 1.2.10 we provide a procedure that may be used to construct a complete measure from an arbitrary measure. Exercises 1.2.4 and 1.2.5 prove that properties of complete measure spaces are different from those of measure spaces that are not complete.

A map f from a measurable space $(\Omega, \mathcal{F})$ to a measurable space $(\Omega', \mathcal{F}')$ is said to be $\mathcal{F}$ **measurable**, or just **measurable** iff for any set $A \in \mathcal{F}'$ the inverse image $f^{-1}(A)$ belongs to $\mathcal{F}$. If, additionally, all inverse images of measurable sets belong to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, then we say that f is $\mathcal{G}$ **measurable**, or more precisely $\mathcal{G}/\mathcal{F}'$ **measurable**. If f is a measurable function from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$ then

$$\sigma_f = \{A \in \mathcal{F} | A = f^{-1}(B) \text{ where } B \in \mathcal{F}'\}$$

is a sub-σ-algebra of $\mathcal{F}$. σ_f is called the σ-algebra **generated by** f. Of course, f is $\mathcal{G}$ measurable if $\sigma_f \subset \mathcal{G}$.

The σ-algebra of Lebesgue measurable subsets of a measurable subset $A \subset \mathbb{R}^n$ is denoted $\mathcal{M}_n(A)$ or $\mathcal{M}(A)$ if n is clear from the context, and the Lebesgue measure in this space is denoted leb_n, or simply leb. A standard result says that $\mathcal{M} := \mathcal{M}(\mathbb{R}^n)$ is the smallest complete σ-algebra containing $\mathcal{B}(\mathbb{R}^n)$. In considering the measures on $\mathbb{R}^n$ we will always assume that they are defined on the σ-algebra of Lebesgue measurable sets, or Borel sets. The interval $[0, 1)$ with the family of its Lebesgue subsets and the Lebesgue measure restricted to these subsets is often referred to as **the standard probability space**. An n-**dimensional random vector** (or simply n-vector) is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. A **complex-valued random variable** is simply a two dimensional random vector; we tend to use the former name if we want to consider complex products of two-dimensional random vectors. Recall that any random n-vector $\underline{X}$ is of the form $\underline{X} = (X_1, ..., X_n)$ where X_i are random variables $X_i : \Omega \to \mathbb{R}$.

1.2.2 **Exercise** Let A be an open set in $\mathbb{R}^n$. Show that A is union of all balls contained in A with rational radii and centers in points with rational coordinates. Conclude that $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by open (resp. closed) intervals. The same result is true for intervals of the form $(a, b]$ and $[a, b)$. Formulate and prove an analog in $\mathbb{R}^n$.

1.2.3 **Exercise** Suppose that Ω and Ω' are topological spaces. If a map $f : \Omega \to \Omega'$ is continuous, then f is measurable with respect to Borel σ-fields in Ω and Ω'. More generally, suppose that f maps a measurable space $(\Omega, \mathcal{F})$ into a measurable space $(\Omega, \mathcal{F}')$, and that $\mathcal{G}'$ is a class of measurable subsets of Ω' such $\sigma(\mathcal{G}') = \mathcal{F}'$. If inverse images of elements of $\mathcal{G}'$ are measurable, then f is measurable.

1.2.4 **Exercise** Suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space, and f maps Ω into $\mathbb{R}$. Equip $\mathbb{R}$ with the σ-algebra of Borel sets and prove that f is measurable iff sets of the form $\{\omega | f(\omega) \leq t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$. (Equivalently: sets of the form $\{\omega | f(\omega) < t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$.) Prove by example that a similar statement is not necessarily true if Borel sets are replaced by Lebesgue measurable sets.

1.2.5 **Exercise** Let $(\Omega, \mathcal{F}, \mu)$ be a *complete* measure space, and f be a map $f : \Omega \to \mathbb{R}$. Equip $\mathbb{R}$ with the algebra of Lebesgue measurable sets and prove that f is measurable iff sets of the form $\{\omega | f(\omega) \leq t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$. (Equivalently: sets of the form $\{\omega | f(\omega) < t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$.)

1.2.6 **Exercise** Let $(S, \mathcal{U})$ be a topological space and let S' be its subset. We can introduce a natural topology in S', termed **induced**

topology, to be the family of sets $U' = U \cap S'$ where U is open in S. Show that

$$\mathcal{B}(S') = \{B \subset S' | B = A \cap S', A \in \mathcal{B}(S)\}. \tag{1.3}$$

1.2.7 *Monotone class theorem* A class $\mathcal{G}$ of subsets of a set Ω is termed a **π-system** if the intersection of any two of its elements belongs to the class. It is termed a **λ-system** if (a) Ω belongs to the class, (b) $A, B \in \mathcal{G}$ and $A \subset B$ implies $B \setminus A \in \mathcal{G}$ and (c) if $A_1, A_2, ... \in \mathcal{G}$, and $A_1 \subset A_2 \subset \cdots$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{G}$. The reader may prove that a λ-system that is at the same time a π-system is also a σ-algebra. In 1.4.3 we exhibit a natural example of a λ-system that is not a σ-algebra. The **Monotone Class Theorem** or **π–λ theorem**, due to W. Sierpiński, says that if $\mathcal{G}$ is a π-system and $\mathcal{F}$ is a λ-system and $\mathcal{G} \subset \mathcal{F}$, then $\sigma(\mathcal{G}) \subset \mathcal{F}$. As a corollary we obtain the uniqueness of extension of a measure defined on a π-system. To be more specific, if $(\Omega, \mathcal{F})$ is a measure space, and $\mathcal{G}$ is a π-system such that $\sigma(\mathcal{G}) = \mathcal{F}$, and if μ and μ' are two finite measures on $(\Omega, \mathcal{F})$ such that $\mu(A) = \mu'(A)$ for all $A \in \mathcal{G}$, then the same relation holds for $A \in \mathcal{F}$. See [5].

1.2.8 *Existence of an extension of a measure* A standard construction involving the so-called outer measure shows the existence of an extension of a measure defined on a field. To be more specific, if μ is a finite measure on a field $\mathcal{F}$, then there exists a measure $\tilde{\mu}$ on $\sigma(\mathcal{F})$ such that $\tilde{\mu}(A) = \mu(A)$ for $A \in \mathcal{F}$, see [5]. It is customary and convenient to omit the "~" and denote both the original measure and its extension by μ. This method allows us in particular to prove existence of the Lebesgue measure [5, 106].

1.2.9 *Two important properties of the Lebesgue measure* An important property of the Lebesgue measure is that it is **regular**, which means that for any Lebesgue measurable set A and $\epsilon > 0$ there exists an open set $G \supset A$ and a compact set $K \subset A$ such that $leb(G \setminus K) < \epsilon$. Also, the Lebesgue measure is **translation invariant**, i.e. $leb\, A = leb\, A_t$ for any Lebesgue measurable set A and $t \in \mathbb{R}$, where

$$A_t = \{s \in \mathbb{R}; s - t \in A\}. \tag{1.4}$$

1.2.10 **Exercise** Let $(\Omega, \mathcal{F})$ be a measure space and μ be a measure, not necessarily complete. Let $\mathcal{F}_0$ be the class of subsets B of Ω such that there exists a $C \in \mathcal{F}$ such that $\mu(C) = 0$ and $B \subset C$. Let $\mathcal{F}_\mu = \sigma(\mathcal{F} \cup \mathcal{F}_0)$. Show that there exists a unique extension of μ to $\mathcal{F}_\mu$, and $(\Omega, \mathcal{F}_\mu, \mu)$ is a complete measure space. Give an example of two Borel measures μ and ν such that $\mathcal{F}_\mu \neq \mathcal{F}_\nu$.

1.2.11 *Integral* Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. The integral $\int f \, d\mu$ of a **simple measurable function** f, i.e. of a function of the form $f = \sum_{i=1}^n c_i 1_{A_i}$ where n is an integer, c_i are real constants, A_i belong to $\mathcal{F}$, and $\mu(A_i) < \infty$, is defined as $\int f \, d\mu = \sum_{i=1}^n c_i \mu(A_i)$. We check that this definition of the integral does not depend on the choice of representation of a simple function. The integral of a non-negative measurable function f is defined as the supremum over integrals of non-negative simple measurable functions f_s such that $f_s \leq f$ (μ a.e.). This last statement means that $f_s(\omega) \leq f(\omega)$ for all $\omega \in \Omega$ outside of a measurable set of μ-measure zero. If this integral is finite, we say that f is **integrable**.

Note that in our definition we may include functions f such that $f(\omega) = \infty$ on a measurable set of ωs. We say that such functions have their values in an extended non-negative half-line. An obvious necessary requirement for such a function to be integrable is that the set where it equals infinity has measure zero (we agree as it is customary in measure theory that $0 \cdot \infty = 0$).

If a measurable function f has the property that both $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are integrable then we say that f is **absolutely integrable** and put $\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$. The reader may check that for a simple function this definition of the integral agrees with the one given initially. The integral of a complex-valued map f is defined as the integral of its real part plus $\imath$ (the imaginary unit) times the integral of its imaginary part, whenever these integrals exist. For any integrable function f and measurable set A the integral $\int_A f \, d\mu$ is defined as $\int 1_A f \, d\mu$.

This definition implies the following elementary estimate which proves useful in practice:

$$\left| \int_A f \, d\mu \right| \leq \int_A |f| \, d\mu. \tag{1.5}$$

Moreover, for any integrable functions f and g and any α and β in $\mathbb{R}$, we have

$$\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu.$$

In integrating functions defined on $(\mathbb{R}^n, \mathcal{M}_n(\mathbb{R}^n), leb_n)$ it is customary to write $\mathrm{d}s_1 ... \mathrm{d}s_n$ instead of $\mathrm{d}\, leb_n(\underline{s})$ where $\underline{s} = (s_1, ..., s_n)$. In one dimension, we write $\mathrm{d}s$ instead of $\mathrm{d}leb(s)$.

There are two important results concerning limits of integrals defined this way that we will use often. The first one is called **Fatou's Lemma** and the second **Lebesgue Dominated Convergence Theorem**. The former says that for a sequence of measurable functions f_n with values in the extended non-negative half-line $\liminf_{n\to\infty} \int f_n \,\mathrm{d}\mu \geq \int \liminf_{n\to\infty} f_n \,\mathrm{d}\mu$, and the latter says that if f_n is a sequence of measurable functions and there exists an integrable function f such that $|f_n| \leq f$ (μ a.e.), then $\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$, provided f_n tends to g pointwise, except perhaps on a set of measure zero. Observe that condition $|f_n| \leq f$ implies that f_n and g are absolutely integrable; the other part of the Lebesgue Dominated Convergence Theorem says that $\int |f_n - g| \,\mathrm{d}\mu$ tends to zero, as $n \to \infty$. The reader may remember that both above results may be derived from the **Monotone Convergence Theorem**, which says that if f_n is a sequence of measurable functions with values in the extended non-negative half-line, and $f_{n+1}(\omega) \geq f_n(\omega)$ for all ω except maybe on a set of measure zero, then $\int_A f_n \,\mathrm{d}\mu$ tends to $\int_A \lim_{n\to\infty} f_n(\omega) \,\mathrm{d}\mu$ regardless of whether the last integral is finite or infinite. Here A is the set where $\lim_{n\to\infty} f_n(\omega)$ exists, and by assumption it is a complement of a set of measure zero.

Note that these theorems are true also when, instead of a sequence of functions, we have a family of functions indexed, say, by real numbers and consider a limit at infinity or at some point of the real line.

1.2.12 **Exercise** Let (a, b) be an interval and let, for τ in this interval, $x(\tau, \omega)$ be a given integrable function on a measure space $(\Omega, \mathcal{F}, \mu)$. Suppose furthermore that for almost all $\omega \in \Omega$, $\tau \to x(\tau, \omega)$ is continuously differentiable and there exists an integrable function y such that $\sup_{\tau\in(a,b)} |x'(\tau, \omega)| \leq y(\omega)$. Prove that $z(\tau) = \int_\Omega x(\tau, \omega)\, \mu(\mathrm{d}\omega)$ is differentiable and that $z'(\tau) = \int_\Omega x'(\tau, \omega)\, \mu(\mathrm{d}\omega)$.

1.2.13 *Product measures* Let $(\Omega, \mathcal{F}, \mu)$ and $(\Omega', \mathcal{F}', \mu')$ be two σ-finite measure spaces. In the Cartesian product $\Omega \times \Omega'$ consider the **rectangles**, i.e. the sets of the form $A \times A'$ where $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$, and the function $\mu \otimes \mu'(A \times A') = \mu(A)\mu'(A')$. Certainly, rectangles form a π-system, say $\mathcal{R}$, and it may be proved that $\mu \otimes \mu'$ is a measure on $\mathcal{R}$ and that there exists an extension of $\mu \otimes \mu'$ to a measure on $\sigma(\mathcal{R})$, which is necessarily unique. This extension is called the **product measure** of μ and μ'. The assumption that μ and μ' are σ-finite