ADVANCED MATHEMATICS IN PHYSICS AND ENGINEERING

BRONWELL

Advanced Mathematics in Physics and Engineering

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PREFACE

At a time when textbooks in mathematics, physics, and engineering are emerging in endless procession, there rests upon the author of a new text an obligation to justify his efforts either in terms of contributions to a better understanding of the subject or to the advancement of the frontiers of knowledge. This text is intended primarily for students in engineering and physics at the senior and graduate level. Its preparation has been guided by the following principles:

a. It endeavors to present a reasonably comprehensive exposition of the branches of advanced mathematics which constitute the principal analytical methods used throughout physics and engineering. It is hoped that this will enable the student to gain broader horizons of knowledge and to acquire a higher degree of mathematical proficiency. To this end, clarity and understanding have been the foremost considerations, although a moderate balance of mathematical rigor has been sought.

b. It is the author's belief that the basic laws in many of the more important areas of physics and engineering can be expressed in very general form by a few fundamental mathematical formulations. These general formulations provide a broad perspective of the physical sciences and form the springboard for the development of vast areas of applications.

In this text an attempt has been made to develop the fundamental formulations in those fields which are the common ground of the physicist and the engineer. It is then shown how these simplify for special cases to the equations which usually form the starting point in the solution of problems in physics and engineering. Solutions of typical problems are included in order to provide concrete examples of the mathematical methods.

Undergraduate courses throughout physics and engineering are characterized by a strong propensity to use simplified mathematical formulations in order to achieve clarity. Often these formulations are special cases of a broad fundamental theory, but in the process of simplification, the fundamental relationships are lost. A course in advanced mathematics in physics and engineering offers a singular opportunity to impart a new dimension of breadth and perspective to the student's understanding of mathematics and the physical sciences. It is the author's opinion that this can be accomplished without devoting an excessive amount of time to the rigors of advanced mathematics. Such an approach enables the student to survey the surrounding terrain from the mountain peaks, rather than become eternally engaged in struggling up the mountainsides from the lower reaches. But an aesthetic appreciation alone from the vantage point of the lofty peaks is not enough. The student must also descend into the valleys and the forests and travel the many interesting trails if he would gain mastery of the subject.

c. Finally, anyone who has studied mathematical methods in physics and engineering cannot help being impressed by the strong underlying unity in the methods of mathematical analysis in many fields of physics and engineering. A course in advanced mathematics offers a unique opportunity to explore this fundamental unity in the mathematical methods. In this text an attempt has been made to develop the mathematical analysis of various fields along similar lines so as to emphasize this unity.

The first five chapters present a mathematical foundation in complex numbers, infinite series, the solution of ordinary differential equations, and series methods of solving differential equations, including the Bessel, Legendre, and associated Legendre equations. There follows a chapter on partial differentiation, which includes as an application some of the fundamental formulations in thermodynamics. The next two chapters are devoted to the analysis of mechanical vibration and electrical oscillation in systems containing lumped and distributed elements. The treatment of systems with distributed elements provides a convenient preview of methods of solving partial differential equations, a subject which is treated with considerably greater generality in Chap. 11. The Lagrangian method of formulating differential equations and its relation to Hamilton's principle and the action integral are considered in Chap. 9.

An introduction to the subject of vector analysis is given in Chap. 10. Here one might question whether the subject of vector analysis or that of functions of a complex variable should be treated first. Both subjects have their own particular areas of application. In dealing with field and flow problems, however, the vector-analysis approach is unquestionably more general. The fundamental physical laws can be derived in very general form using the vector-analysis method of expression. These relationships can be readily specialized for problems in one, two, or three dimensions in any orthogonal coordinate system. The complex-variable method, on the other hand, is of little use in the derivation of the general formulations for field and flow problems. Furthermore, its use is restricted to a rather narrow range of problems which satisfy Laplace's equation and which can be expressed in either one or two dimensions. For example, many of the interesting and useful solutions of the wave equations in Chap. 11 can be obtained by the methods of vector analysis, but not by those of the complex variable.

Chapter 11 is an innovation in texts of this nature. It presents a general treatment of the solution of the wave equations, Laplace's equation, the heat-flow equation, the chemical-diffusion equation and other linear partial differential equations. Because these fundamental equations pervade all branches of physics and engineering, an attempt has been made to present a simple and unified approach to their solution. The author has always been somewhat perturbed at the haphazard way in which this subject is usually handled. The customary approach is to present solutions of a multiplicity of isolated problems. After obtaining many seemingly unrelated solutions, the student, in time, acquires experience in the types of solutions which he is likely to encounter.

The approach used in this text is more general and is based upon obtaining a common solution, applicable to all the foregoing equations, in the form of characteristic functions expressed in rectangular, cylindrical, and spherical coordinates. Once these solutions have been obtained, it is a relatively simple matter to select the proper characteristic functions for a particular static or dynamic problem in either one, two, or three dimensions. It is hoped that this approach will provide unity and perspective in the solution of what probably constitutes the most important segment of mathematical analysis throughout all of physics and engineering.

The fundamentals of heat flow, fluid dynamics, and electromagnetic theory occupy Chaps. 12, 13, and 14. In the chapter on fluid dynamics all equations for both compressible and incompressible fluid flow are derived from three fundamental equations, namely, Euler's equation, the equation of continuity, and the equation of state. In the chapter on electromagnetic theory, all equations are derived from Maxwell's equations.

Chapter 15 is devoted to the functions of a complex variable, including Cauchy s theorems, methods of contour integration, and conformal transformation. The theory of dynamic stability of airplanes, servomechanisms, and electrical networks provides an interesting and useful application of the theory of functions of the complex variable. This subject is of importance to the engineer and the physicist alike, and it is one in which the technical literature is entirely inadequate but growing rapidly. The proofs of the Routh-Hurwitz stability criterion, the Nyquist criterion, and other stability criteria are presented, together with applications, in Chap. 16.

The final chapter presents an introduction to Laplacian methods in operational calculus. This subject, which started out on an insecure mathematical footing, has now taken its rightful place as a powerful field of mathematical analysis, applicable to a wide area of problems in physics and engineering.

Throughout the text, emphasis has been placed upon applications in dynamics, rather than in statics. This is a recognition of the fact that most of the interesting and useful problems in physics and engineering are basically dynamic problems. Also, experience has shown that a student who has mastered the analysis of problems in dynamics usually experiences little difficulty in solving comparable problems involving statics, whereas the converse is not often true.

The problems have been carefully chosen to supplement the text material with additional information. Perhaps it is appropriate to inform the student that some of the problems in this book appear as text developments in the reference books which are listed at the end of each chapter. If this provides the incentive for the student to seek out and cultivate an acquaintanceship with the many excellent source books, its mission will be amply fulfilled.

The material has been arranged with a view toward achieving an expeditious coverage of the essential subjects. The first few chapters provide background material which the students will probably have had in previous courses. An undergraduate course, following a course in differential equations, might include Chaps. 5, 6, 7, 8, 10, 11, and the first parts of Chaps. 12, 13, and 14. It will probably be found expedient to include only part of the material in Chap. 5, since this subject has been treated rather thoroughly for reference purposes. An alternative arrangement would be to take up the applications of differential equations in Chaps. 7 and 8 (except for Art. 8.15) before treating Chap. 5. If it is desired to use the Laplace transform method for the solution of differential equations, Chap. 17 can be introduced early in the course, since most of this chapter is not dependent upon the preceding chapters.

The more fundamental material and the simpler applications have been placed toward the beginning of each chapter, with the material increasing in complexity as the subject is developed. This permits any desired depth of penetration without the necessity of covering all the material.

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CONTENTS

CHAPTER 1

INFINITE SERIES

Series methods play an important role in the mathematics of science and engineering. In many problems, the application of series methods facilitates solutions which would be difficult or impossible to obtain by other means. The general solution of certain types of differential equations, such as the Bessel equation and the Legendre equation, can be obtained only by the use of infinite series. These solutions, in series form, are used to define new functions, known as Bessel functions, Legendre functions, etc.

In the use of an infinite series, it is necessary to know whether or not the series converges to a unique limiting value. A given series may converge for certain values of the variable and diverge for other values; hence we define the *region* or *interval of convergence* as the range of values of the variable for which the series converges. There are many convergence tests which can be used to test a series for convergence. We shall consider several of the more useful tests. In the interest of brevity, proofs of some of the theorems have been omitted.

The terms of a series may be functions of either a real variable or a complex variable. In this chapter, we shall consider only series containing real variables. However, it is well to remember that many of the convergence tests for series of real variables can be adapted to series containing complex variables. By separating real and imaginary parts of a complex series and equating reals and imaginaries on both sides of the equation, it is often possible to reduce a single complex series to two series, one containing the real terms and the other containing the imaginary terms. Each of these series can be tested for convergence by the methods of this chapter. If both series converge for a given value of the complex variable, then the complex series converges. If either series diverges, then the complex series diverges.

The interval of convergence of a series containing a real variable can be represented on the real axis of that variable. In the case of the complex series, we speak of the region or domain of convergence.

1.1. Infinite Series. A series may contain either a finite number of terms or an infinite number of terms. Thus, the series containing functions of a real variable x, represented by

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots = \sum_{n=1}^{\infty} u_n(x) \tag{1}$$

is an infinite series. If the series is evaluated for a fixed value of the variable $x = x_0$, there results a series of constant terms. This series converges if the sum of the terms of the series approaches a unique limiting value as the number of terms increases indefinitely. Thus, if S_n is the *partial sum* of the first n terms, then the series converges if

$$\lim_{n\to\infty} S_n(x_0) = S(x_0) \tag{2}$$

exists, where $S(x_0)$ is a constant defined as the sum of the series.

A necessary condition for convergence is that the nth term of the series approach zero in the limit. Thus, the series of constant terms

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

cannot converge unless

$$\lim_{n\to\infty} a_n = 0 \tag{3}$$

This is a *necessary* but not a *sufficient* condition for convergence. There are many series which satisfy (3) but which do not converge. For example, in the *harmonic series*

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} \tag{4}$$

we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 0$. The series, however, diverges as is evident from the fact that we can regroup the terms

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots$$

such that each term in parenthesis is greater than $\frac{1}{2}$. The series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ clearly diverges; hence by comparison, the harmonic series likewise diverges.

A more precise definition of convergence may be stated as follows: *A series* $\sum_{n=1}^{\infty} u_n(x)$ *converges for a particular value of the variable* $x = x_0$ *if, for a given positive number* ϵ, *which can be arbitrarily small, there exists a positive integer* N *such that for all integer values of* $n > N$

$$| S_n(x_0) - S_N(x_0) | < \epsilon \tag{5}$$

The value of N will be dependent upon the choice of ϵ.

In a series which diverges, the partial sum of the first n terms may either approach infinity as $n \to \infty$, or it may oscillate without approaching a limit.

1.2. Comparison Test for Convergence. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series of finite positive constants which is to be tested for convergence. For comparison, let* $\sum_{n=1}^{\infty} c_n$ *be a series of positive constants which is known to converge and* $\sum_{n=1}^{\infty} d_n$ *be* *a series of positive constants which is known to diverge. If $a_n \leqq c_n$ for all values of $n > N$, where N is an arbitrary positive integer, then the $\sum a_n$ series converges. If $a_n \geqq d_n$ for all values of $n > N$, then the $\sum a_n$ series diverges.*

To prove the convergence portion of this theorem, we note that the first N terms of the $\sum a_n$ series have a finite sum. Since the remaining terms from $n = N$ to $n = \infty$ are term for term less than those of the $\sum c_n$ series, the sum of these terms must clearly be less than the sum of the corresponding terms of the $\sum c_n$ series. Since, by hypothesis, the sum of the terms in the $\sum c_n$ series approaches a limit as $n \to \infty$, it follows that the sum of the terms in the $\sum a_n$ series must likewise approach a limit as $n \to \infty$; hence the $\sum a_n$ series converges. A similar proof can be established for the divergence criterion.

The p series and the geometric series are useful in making comparison tests. The p series is

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1}$$

This will later be shown to converge for $p > 1$ and to diverge for $p \leqq 1$.

The geometric series

$$a(1 + r + r^2 + \cdots) = \sum_{n=0}^{\infty} ar^n \tag{2}$$

converges to $S = a/(1 - r)$ when $|r| < 1$ and diverges if $|r| \geqq 1$. As an example of the comparison test, consider the series

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \tag{3}$$

Each term of the series is less than a corresponding term of the p series with $p = 2$, which is known to converge; hence (3) converges.

1.3. Absolute Convergence. *If, in a series of positive and negative terms, each term is replaced by its absolute value and the resulting series converges in a given interval, then the original series converges and is said to converge absolutely in that interval.*

If the series of absolute values diverges, the original series may either converge or diverge; but if it converges, it is said to *converge conditionally*.

For example, if the series

$$u_1(x_0) + u_2(x_0) + u_3(x_0) + \cdots = \sum_{n=1}^{\infty} u_n(x_0) \tag{1}$$

contains both positive and negative terms, we write a new series of absolute values of (1), thus

$$|u_1(x_0)| + |u_2(x_0)| + |u_3(x_0)| + \cdots = \sum_{n=1}^{\infty} |u_n(x_0)| \tag{2}$$

If series (2) converges, then (1) is *absolutely convergent*.

To prove the theorem, let $S_n^{(+)}(x_0)$ be the partial sum of the positive terms and $-S_n^{(-)}(x_0)$ be the partial sum of the negative terms up to and including the nth term. The sum of the first n terms of series (1) is then

$$S_{n1}(x_0) = S_n^{(+)}(x_0) - S_n^{(-)}(x_0)$$

The sum of the first n terms of (2) is

$$S_{n2}(x_0) = S_n^{(+)}(x_0) + S_n^{(-)}(x_0)$$

Since $S_n^{(+)}$ and $S_n^{(-)}$ are both positive numbers, it is clear that

$$\lim_{n\to\infty} S_{n2}(x_0) > \lim_{n\to\infty} S_{n1}(x_0)$$

assuming that both limits exist. Consequently, if (2) converges, (1) must likewise converge and its sum must be less than the sum of series (2).

1.4. Ratio Test for Convergence. The ratio test is a simple and extremely useful convergence test. It may be stated as follows:

A series of constants $\sum_{n=1}^{\infty} a_n$ *converges absolutely if*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

The series diverges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

The series may either converge or diverge if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

The theorem may be proved by use of the comparison test. Consider first the series of absolute values $\sum_{n=1}^{\infty} |a_n|$. In this series, let it be assumed that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Now choose a number r, such that $L < r < 1$. There will then be a value of $n = N$, such that

$$\left| \frac{a_{N+1}}{a_N} \right| < r \qquad \left| \frac{a_{N+2}}{a_{N+1}} \right| < r \qquad \left| \frac{a_{N+3}}{a_{N+2}} \right| < r \qquad \text{etc.}$$

Therefore

$$|a_{N+1}| < r\,|a_N|$$

$$|a_{N+2}| < r\,|a_{N+1}| < r^2\,|a_N|$$

$$|a_{N+3}| < r\,|a_{N+2}| < r^3\,|a_N|$$

The sum of the terms of the series of absolute values from $n = N$ to $n = \infty$ is therefore

$$|a_N| + |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots < (1 + r + r^2 + r^3 + \cdots)\, |a_N|$$

The series on the right is the geometric series which converges, since by hypothesis $r < 1$. Hence, the series on the left likewise converges. We conclude, therefore, that, if $L < 1$, the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges. Finally, by Art. 1.3, if the series of absolute values converges, the original series $\sum_{n=1}^{\infty} a_n$, which may contain both positive and negative terms, converges absolutely.

As an example, let us test the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{k^{2n-1}}{(n+1)!} \tag{1}$$

The ratio test yields

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(k^{2n+1})(n+1)!}{(n+2)!(k^{2n-1})} \right|$$

which may be written

$$\lim_{n\to\infty} \left| \frac{k^2(k^{2n-1})(n+1)!}{(n+2)(n+1)!k^{2n-1}} \right| = \lim_{n\to\infty} \left| \frac{k^2}{n+2} \right| = 0$$

Hence, by the ratio test, the series converges for all finite values of k.

When we attempt to apply the ratio test to the p series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p} \tag{2}$$

we obtain

$$\lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^p} = 1$$

hence the ratio test fails.

However, if we let $p = 1$, series (2) reduces to the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which was shown in Art. 1.1 to diverge. When $p < 1$, each term of series (2) is greater than the corresponding term of the harmonic series; hence by the comparison test, the p series diverges for $p \leqq 1$.

Now consider $p > 1$. For comparison, we use the series

$$1 + \frac{1}{2^{k-1}} + \frac{1}{(2^{k-1})^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2^{k-1})^n} \tag{3}$$