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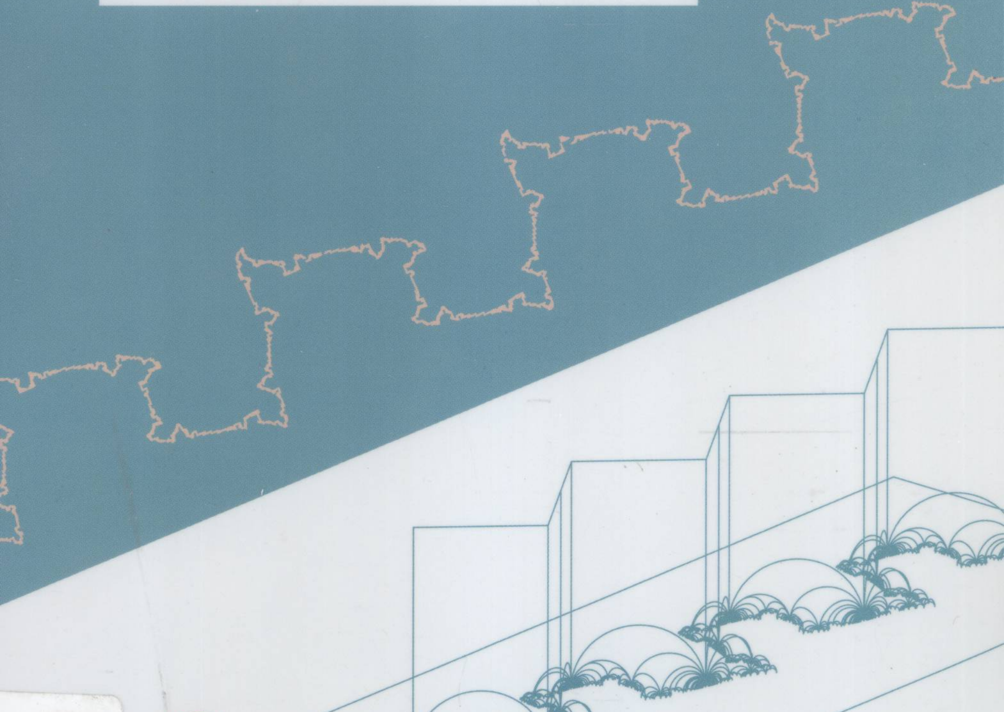
Low-Dimensional Geometry

From Euclidean Surfaces to Hyperbolic Knots

Francis Bonahon



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Low-Dimensional Geometry

From Euclidean Surfaces to Hyperbolic Knots

à la souris et à l'écureuil

IAS/Park City Mathematics Institute

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the "Regional Geometry Institute" initiative of the National Science Foundation. In mid-1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey. The PCMI continues to hold summer programs in Park City, Utah.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, mathematics education researchers, and undergraduate faculty. One of PCMI's main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at sites around the country form an integral part of the High School Teacher Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Program deal with this topic as well. Lecture notes from the Graduate Summer School are published each year in the IAS/Park City Mathematics Series. Course materials from the Undergraduate Program, such as the current volume, are now being published as part of the IAS/Park City Mathematical Subseries in the Student Mathematical Library. We are happy to make available more of the excellent resources which have been developed as part of the PCMI.

John Polking, Series Editor
April 13, 2009

Preface

About 30 years ago, the field of 3-dimensional topology was revolutionized by Thurston's Geometrization Theorem and by the unexpected appearance of hyperbolic geometry in purely topological problems. This book aims at introducing undergraduate students to some of these striking developments. It grew out of notes prepared by the author for a three-week course for undergraduates that he taught at the Park City Mathematical Institute in June–July 2006. It covers much more material than these lectures, but the written version intends to preserve the overall spirit of the course. The ultimate goal, attained in the last chapter, is to bring the students to a level where they can understand the statements of Thurston's Geometrization Theorem for knot complements and, more generally, of the general Geometrization Theorem for 3-dimensional manifolds recently proved by G. Perelman. Another leading theme is the intrinsic beauty of some of the mathematical objects involved, not just mathematically but visually as well.

The first two-thirds of the book are devoted to 2-dimensional geometry. After a brief discussion of the geometry of the euclidean plane $\mathbb{R}^2$, the hyperbolic plane $\mathbb{H}^2$, and the sphere $\mathbb{S}^2$, we discuss the construction of locally homogeneous spaces by gluing the sides of a polygon. This leads to the investigation of the tessellations that are associated to such constructions, with a special focus on one of the most beautiful objects of mathematics, the Farey tessellation of the hyperbolic plane. At this point, the deformations of the Farey tessellation by shearing lead us to jump to one dimension higher, in order to allow bending. After a few generalities on the 3-dimensional hyperbolic space $\mathbb{H}^3$, we consider the crooked tessellations obtained by bending the Farey tessellation, which naturally leads us to discussing kleinian groups and quasi-fuchsian groups. Pushing the bending of the Farey tessellation to the edge of kleinian groups, we reach the famous example associated to the complement of the figure-eight knot. At this point, we are ready to explain that this example is a manifestation of a general phenomenon. We state Thurston's Geometrization Theorem for knot complements, and illustrate how it has revolutionized knot theory in particular through the use of Ford domains. The book concludes with a discussion of the very recently proved Geometrization Theorem for 3-dimensional manifolds.

We tried to strike a balance between mathematical intuition and rigor. Much of the material is unapologetically "picture driven", as we intended to share our own enthusiasm for the beauty of some of the mathematical objects involved. However, we did not want to sacrifice the other foundation of mathematics, namely, the level of certainty provided by careful mathematical proofs. One drawback of this compromise is that the exposition is occasionally interrupted with a few proofs which are more lengthy than difficult, but can somewhat break the flow of the discourse. When this occurs, the reader is encouraged, on a preliminary reading, to first glance at the executive summary of the argument that is usually present at its beginning, and then to grab the remote control and press the "fast forward" button until the first occurrence of the closing symbol . The reader may later need to return to some of the parts that have thus been zapped through, for the sake of mathematical rigor or because subsequent parts of the book may refer to specific arguments or definitions in these sections. For the same reason, the book is not intended to be read in a linear way. The reader is strongly advised to generously skip, at first, much of the early material in order to reach the parts with pretty pictures, such as Chapters 5, 6, 8, 10 or 11, as quickly as possible, and then to backtrack when specific definitions or arguments are needed.

The book also has its idiosyncrasies. From a mathematical point of view, the main one involves quotients of metric spaces. It is traditional here to focus only on topological spaces, to introduce the quotient topology by *fiat*, and then to claim that it accurately describes the intuitive notion of gluing in cut-and-paste constructions; this is not always very convincing. A slightly less well-trodden road involves quotient metric spaces, but only in the case of quotients under discontinuous group actions. We decided to follow a different strategy, by discussing quotient (semi-)metrics very early on and in their full generality. This approach is, in our view, much more intuitive but it comes with a price: Some proofs become somewhat technical. On the one hand, these can serve as a good introduction to the techniques of rigorous proofs in mathematics. On the other hand, the reader pressed for time can also take advantage of the fast-forward commands where indicated, and zap through these proofs in a first reading.

From a purely technical point of view, the text is written in such a way that, in theory, it does not require much mathematical knowledge beyond multivariable calculus. An appendix at the end provides a "tool kit" summarizing some of the main concepts that will be needed. In practice, however, the mathematical rigor of many arguments is likely to require a somewhat higher level of mathematical sophistication. The reader will also notice that the level of difficulty progressively increases as one proceeds from early to later chapters. Each chapter ends with a selection of exercises, a few of which can be somewhat challenging. The idea was to provide material suitable for an independent study by a dedicated undergraduate student, or for a topics course. Such a course might cover the main sections of Chapters 1–7, 9, 12, and whichever parts of the remaining chapters would be suitable for both the time available and the tastes of the instructor.

The author is delighted to thank Roger Howe for tricking him into believing that the PCMI course would not require that much work (which turned out to be wrong), and Ed Dunne for encouraging him to turn the original lecture notes into a book and for warning him that the task would be very labor intensive (which turned out to be right). The general form of the book owes much to the feedback received from the students and faculty who attended the PCMI lectures, and who were used as "guinea pigs"; this includes Chris Hiatt, who was the teaching assistant for the course. Dave Futer provided numerous and invaluable comments on an earlier draft of the manuscript, Roland van der Veen contributed a few more, and Jennifer Wright Sharp polished the final version with her excellent copy-editing. Finally, the mathematical content of the book was greatly influenced by the author's own research in this area of mathematics, which in recent years was partially supported by Grants 0103511 and 0604866 from the National Science Foundation.

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Chapter 1

The euclidean plane

We are all very familiar with the geometry of the euclidean plane $\mathbb{R}^2$. We will encounter a new type of 2-dimensional geometry in the next chapter, that of the hyperbolic plane $\mathbb{H}^2$. In this chapter, we first list a series of well-known properties of the euclidean plane which, in the next chapter, will enable us to develop the properties of the hyperbolic plane in very close analogy.

Before proceeding, you are advised to briefly consult the TOOL KIT in the appendix for a succinct review of the basic definitions and notation concerning set theory, infima and suprema of sets of real numbers, and complex numbers.

1.1. Euclidean length and distance

The ***euclidean plane*** is the set

$$\mathbb{R}^2 = \{(x, y); x, y \in \mathbb{R}\}$$

consisting of all ordered pairs (x, y) of real numbers x and y.

If γ is a curve in $\mathbb{R}^2$, parametrized by the differentiable vector-valued function

$$t \mapsto \big(x(t), y(t)\big), \quad a \leqslant t \leqslant b,$$

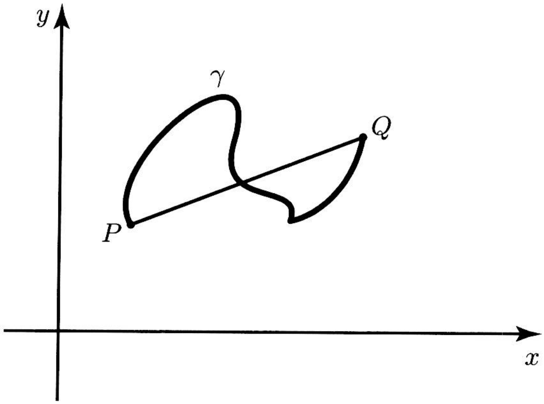

Figure 1.1. The euclidean plane

its ***euclidean length*** $\ell_{\text{euc}}(\gamma)$ is the arc length given by

$$\ell_{\text{euc}}(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt. \tag{1.1}$$

This length is independent of the parametrization by a well-known consequence of the chain rule.

It will be convenient to consider ***piecewise differentiable*** curves γ made up of finitely many differentiable curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that the initial point of each γ_{i+1} is equal to the terminal point of γ_i. In other words, such a curve γ is differentiable everywhere except at finitely many points, corresponding to the endpoints of the γ_i, where it is allowed to have a "corner" (but no discontinuity). In this case, the length $\ell_{\text{euc}}(\gamma)$ of the piecewise differentiable curve γ is defined as the sum of the lengths $\ell_{\text{euc}}(\gamma_i)$ of its differentiable pieces γ_i. This is equivalent to allowing the integrand in (1.1) to be undefined at finitely many values of t where, however, it has finite left-hand and right-hand limits.

The ***euclidean distance*** $d_{\text{euc}}(P, Q)$ between two points P and Q is the infimum of the lengths of all piecewise differentiable curves γ going from P to Q, namely

$$d_{\text{euc}}(P, Q) = \inf\left\{\ell_{\text{euc}}(\gamma); \gamma \text{ goes from } P \text{ to } Q\right\}. \tag{1.2}$$