METHODS OF MODERN MATHEMATICAL PHYSICS

II

Fourier Analysis, Self-Adjointness

现代数学物理方法

第2卷

Michael Reed / Barry Simon

ELSEVIER

Elsevier (Singapore) Pte Ltd.

世界图书出版公司

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MODERN MATHEMATICAL PHYSICS

II: FOURIER ANALYSIS, SELF-ADJOINTNESS

MICHAEL REED

Department of Mathematics
Duke University

BARRY SIMON

Departments of Mathematics
and Physics
Princeton University

Academic Press
San Diego New York Boston
London Sydney Tokyo Toronto

书　　名：Methods of Modern Mathematical Physics II
作　　者：M. Reed, B. Simon
中 译 名：现代数学物理方法 第 2 卷
出 版 者：世界图书出版公司北京公司
印 刷 者：北京世图印刷厂
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：010-64015659, 64038347
电子信箱：kjsk@vip.sina.com
开　　本：24　　　印　张：16
出版年代：2003 年 6 月
书　　号：7-5062-5932-X/ O · 351
版权登记：图字: 01-2003-4002
定　　价：53.00 元

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Methods of Modern Mathematical Physics
Vol. 2:Fourier Analysis, Self-Adjointness
Michael Reed, Barry Simon
ISBN: 0-12-585002-6

Authorized English language reprint edition published by the Proprietor.
Reprint ISBN: 981-4141-66-6

Elsevier (Singapore) Pte Ltd.
3 Killiney Road
#08-01 Winsland Hose I
Sinagpore 239519
Tel: (65) 6349-0200
Fax: (65) 6733-1817

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To our parents
Helen and Gerald Reed
Minnie and Hy Simon

Preface

This volume continues our series of texts devoted to functional analysis methods in mathematical physics. In Volume I we announced a table of contents for Volume II. However, in the preparation of the material it became clear that we would be unable to treat the subject matter in sufficient depth in one volume. Thus, the volume contains Chapters IX and X; we expect that a third volume will appear in the near future containing the rest of the material announced as "Analysis of Operators." We hope to continue this series with an additional volume on algebraic methods.

It gives us pleasure to thank many individuals:

E. Nelson for a critical reading of Chapter X; W. Beckner, H. Kalf, R. S. Phillips, and A. S. Wightman for critically reading one or more sections.

Numerous other colleagues for contributing valuable suggestions.

F. Armstrong for typing most of the preliminary manuscript.

J. Hagadorn, R. Israel, and R. Wolpert for helping us with the proof-reading.

Academic Press for its aid and patience; the National Science and Alfred P. Sloan Foundations for financial support.

Jackie and Martha for their encouragement and understanding.

MIKE REED
BARRY SIMON

June 1975

Introduction

A functional analyst is an analyst, first and foremost, and not a degenerate species of topologist.
E. Hille

Most texts in functional analysis suffer from a serious defect that is shared to an extent by Volume I of Methods of Modern Mathematical Physics. Namely, the subject is presented as an abstract, elegant corpus generally divorced from applications. Consequently, the students who learn from these texts are ignorant of the fact that almost all deep ideas in functional analysis have their *immediate* roots in "applications," either to classical areas of analysis such as harmonic analysis or partial differential equations, or to another science, primarily physics. For example, it was classical electromagnetic potential theory that motivated Fredholm's work on integral equations and thereby the work of Hilbert, Schmidt, Weyl, and Riesz on the abstractions of Hilbert space and compact operator theory. And it was the impetus of quantum mechanics that led von Neumann to his development of unbounded operators and later to his work on operator algebras.

More deleterious than historical ignorance is the fact that students are too often misled into believing that the most profitable directions for research in functional analysis are the abstract ones. In our opinion, exactly the opposite is true. We do not mean to imply that abstraction has no role to play. Indeed, it has the critical role of taking an idea from a concrete situation and, by eliminating the extraneous notions, making the idea more easily understood as well as applicable to a broader range of situations. But it is the study of specific applications and the consequent generalizations that have been the more important, rather than the consideration of abstract questions about abstract objects for their own sake.

This volume contains a mixture of abstract results and applications, while the next contains mainly applications. The intention is to offer the readers of the whole series a properly balanced view.

We hope that this volume will serve several purposes: to provide an introduction for graduate students not previously acquainted with the material, to serve as a reference for mathematical physicists already working in the field, and to provide an introduction to various advanced topics which are difficult to understand in the literature. Not all the techniques and applications are treated in the same depth. In general, we give a very thorough discussion of the mathematical techniques and applications in quantum mechanics, but provide only an introduction to the problems arising in quantum field theory, classical mechanics, and partial differential equations. Finally, some of the material developed in this volume will not find application until Volume III. For all these reasons, this volume contains a great variety of subject matter. To help the reader select which material is important for him, we have provided a "Reader's Guide" at the end of each chapter.

As in Volume I, each chapter contains a section of notes. The notes give references to the literature and sometimes extend the discussion in the text. Historical comments are always limited by the knowledge and prejudices of authors, but in mathematics that arises directly from applications, the problem of assigning credit is especially difficult. Typically, the history is in two stages: first a specific method (typically difficult, computational, and sometimes nonrigorous) is developed to handle a small class of problems. Later it is recognized that the method contains ideas which can be used to treat other problems, so the study of the method itself becomes important. The ideas are then abstracted, studied on the abstract level, and the techniques systematized. With the newly developed machinery the original problem becomes an easy special case. In such a situation, it is often not completely clear how many of the mathematical ideas were already contained in the original work. Further, how one assigns credit may depend on whether one first learned the technique in the old computational way or in the new easier but more abstract way. In such situations, we hope that the reader will treat the notes as an introduction to the literature and not as a judgment of the historical value of the contributions in the papers cited.

Each chapter ends with a set of problems. As in Volume I, parts of proofs are occasionally left to the problems to encourage the reader to participate in the development of the mathematics. Problems that fill gaps in the text are marked with a dagger. Difficult problems are marked with an asterisk. We strongly urge students to work the problems since that is the best way to learn mathematics.

METHODS OF MODERN MATHEMATICAL PHYSICS

II: FOURIER ANALYSIS, SELF-ADJOINTNESS

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We have therefore the equation of condition

$$F(x) = \int dq\, Q \cos qx$$

If we substituted for Q any function of q, and conducted the integration from $q = 0$ to $q = \infty$, we should find a function of x: it is required to solve the inverse problem, that is to say, to ascertain what function of q, after being substituted for Q, gives as a result the function F(x), a remarkable problem whose solution demands attentive examination. *Joseph Fourier*

IX.1 The Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, convolutions

The Fourier transform is an important tool of both classical and modern analysis. We begin by defining it, and the inverse transform, on $\mathscr{S}(\mathbb{R}^n)$, the Schwartz space of C^∞ functions of rapid decrease.

Definition Suppose $f \in \mathscr{S}(\mathbb{R}^n)$. The **Fourier transform** of f is the function $\hat{f}$ given by

$$\hat{f}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\boldsymbol{\lambda}} f(\mathbf{x})\, d\mathbf{x}$$

where $\mathbf{x} \cdot \boldsymbol{\lambda} = \sum_{i=1}^{n} x_i \lambda_i$. The **inverse Fourier transform** of f, denoted by $\check{f}$, is the function

$$\check{f}(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\boldsymbol{\lambda}} f(\mathbf{x})\, d\mathbf{x}$$

We will occasionally write $\hat{f} = \mathscr{F}f$.

Since every function in Schwartz space is in $L^1(\mathbb{R}^n)$, the above integrals make sense. Many authors begin by discussing the Fourier transform on $L^1(\mathbb{R}^n)$. We start with Schwartz space for two reasons: First, the Fourier transform is a one-to-one map of Schwartz space onto itself (Theorem IX.1). This makes it particularly easy to talk about the inverse Fourier transform, which of course turns out to be the inverse map. That is, on Schwartz space, it is possible to deal with the transform and the inverse transform on an equal footing. Though this is also true for the Fourier transform on $L^2(\mathbb{R}^n)$ (see Theorem IX.6), it is not possible to define the Fourier transform on $L^2(\mathbb{R}^n)$ directly by the integral formula since $L^2(\mathbb{R}^n)$ functions may not be in $L^1(\mathbb{R}^n)$; a limiting procedure must be used. Secondly, once we know that the Fourier transform is a one-to-one, bounded map of $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$, we can easily extend it to $\mathscr{S}'(\mathbb{R}^n)$. It is this extension that is fundamental to the applications in Sections 5, 6, and 8.

We will use the standard multi-index notation. A multi-index

$$\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$$

is an n-tuple of nonnegative integers. The collection of all multi-indices will be denoted by I_+^n. The symbols $|\alpha|$, x^α, D^α, and x^2 are defined as follows:

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

$$x^2 = \sum_{i=1}^{n} x_i^2$$

In preparation for the proof that $\hat{}$ and $\check{}$ are inverses, we prove:

Lemma The maps $\hat{}$ and $\check{}$ are continuous linear transformations of $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$. Furthermore, if α and β are multi-indices, then

$$((i\lambda)^\alpha D^\beta \hat{f})(\lambda) = \widehat{D^\alpha((-ix)^\beta f(x))} \tag{IX.1}$$

Proof The map $\hat{}$ is clearly linear. Since

$$\begin{aligned}(\lambda^\alpha D^\beta \hat{f})(\lambda) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \lambda^\alpha (-ix)^\beta e^{-i\lambda\cdot x} f(x)\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{1}{(-i)^\alpha} (D_x^\alpha e^{-i\lambda\cdot x})(-ix)^\beta f(x)\, dx \\ &= \frac{(-i)^\alpha}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\lambda\cdot x} D_x^\alpha((-ix)^\beta f(x))\, dx\end{aligned}$$

We conclude that

$$\|\hat{f}\|_{\alpha,\beta} = \sup_{\lambda} |\lambda^\alpha (D^\beta \hat{f})(\lambda)| \le \frac{1}{(2\pi)^{n/2}} \int |D_x^\alpha (x^\beta f)| \, dx < \infty$$

so $\hat{}$ takes $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, and we have also proven (IX.1). Furthermore, if k is large enough, $\int (1 + x^2)^{-k} \, dx < \infty$ so that

$$\|\hat{f}\|_{\alpha,\beta} \le \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{(1 + x^2)^{-k}}{(1 + x^2)^{-k}} |D_x^\alpha (-ix)^\beta f(x)| \, dx$$

$$\le \frac{1}{(2\pi)^{n/2}} \left(\int (1 + x^2)^{-k} \, dx \right) \sup_x \{(1 + x^2)^{+k} |D_x^\alpha (-ix)^\beta f(x)|\}$$

Using Leibnitz's rule we easily conclude that there exist multi-indices α_j, β_j and constants c_j so that

$$\|\hat{f}\|_{\alpha,\beta} \le \sum_{i=1}^{M} c_j \|f\|_{\alpha_j, \beta_j}$$

Thus, $\hat{}$ is bounded and by Theorem V.4 is therefore continuous. The proof for $\check{}$ is the same. ∎

We are now ready to prove the Fourier inversion theorem. The proof we give uses the original idea of Fourier.

Theorem IX.1 (Fourier inversion theorem) The Fourier transform is a linear bicontinuous bijection from $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$. Its inverse map is the inverse Fourier transform, i.e., $\check{\hat{f}} = f = \hat{\check{f}}$.

Proof We will prove that $\check{\hat{f}} = f$. The proof that $\hat{\check{f}} = f$ is similar. $\hat{\check{f}} = f$ implies that $\hat{}$ is surjective and $\check{\hat{f}} = f$ implies that $\hat{}$ is injective. Since $\hat{}$ and $\check{}$ are continuous maps of $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, it is sufficient to prove that $\check{\hat{f}} = f$ for f contained in the dense set $C_0^\infty(\mathbb{R}^n)$. Let C_ε be the cube of volume $(2/\varepsilon)^n$ centered at the origin in $\mathbb{R}^n$. Choose ε small enough so that the support of f is contained in C_ε. Let

$$K_\varepsilon = \{\mathbf{k} \in \mathbb{R} \,|\, \text{each } k_i/\pi\varepsilon \text{ is an integer}\}$$

Then

$$f(x) = \sum_{k \in K_\varepsilon} ((\tfrac{1}{2}\varepsilon)^{n/2} e^{ik\cdot x}, f)(\tfrac{1}{2}\varepsilon)^{n/2} e^{ik\cdot x}$$

is just the Fourier series of f which converges uniformly in C_ε to f since f is continuously differentiable (Theorem II.8). Thus

$$f(x) = \sum_{k \in K_\varepsilon} \frac{\hat{f}(k) e^{ik\cdot x}}{(2\pi)^{n/2}} (\pi\varepsilon)^n \tag{IX.2}$$