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Applied Analysis for Engineering Sciences

工程科学中的应用分析

唐少强　编著

北京大学出版社
PEKING UNIVERSITY PRESS

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序　言

物理学是研究物质、能量以及它们之间相互作用的科学。她不仅是化学、生命、材料、信息、能源和环境等相关学科的基础,同时还是许多新兴学科和交叉学科的前沿。在科技发展日新月异和国际竞争日趋激烈的今天,物理学不仅囿于基础科学和技术应用研究的范畴,而且在社会发展与人类进步的历史进程中发挥着越来越关键的作用。

我们欣喜地看到,改革开放三十多年来,随着中国政治、经济、教育、文化等领域各项事业的持续稳定发展,我国物理学取得了跨越式的进步,做出了很多为世界瞩目的研究成果。今日的中国物理正在经历一个历史上少有的黄金时代。

在我国物理学科快速发展的背景下,近年来物理学相关书籍也呈现百花齐放的良好态势,在知识传承、学术交流、人才培养等方面发挥着无可替代的作用。从另一方面看,尽管国内各出版社相继推出了一些质量很高的物理教材和图书,但系统总结物理学各门类知识和发展,深入浅出地介绍其与现代科学技术之间的渊源,并针对不同层次的读者提供有价值的教材和研究参考,仍是我国科学传播与出版界面临的一个极富挑战性的课题。

为有力推动我国物理学研究、加快相关学科的建设与发展,特别是展现近年来中国物理学者的研究水平和成果,北京大学出版社在国家出版基金的支持下推出了"中外物理学精品书系",试图对以上难题进行大胆的尝试和探索。该书系编委会集结了数十位来自内地和香港顶尖高校及科研院所的知名专家学者。他们都是目前该领域十分活跃的专家,确保了整套丛书的权威性和前瞻性。

这套书系内容丰富,涵盖面广,可读性强,其中既有对我国传统物理学发展的梳理和总结,也有对正在蓬勃发展的物理学前沿的全面展示;既引进和介绍了世界物理学研究的发展动态,也面向国际主流领域传播中国物理的优秀专著。可以说,"中外物理学精品书系"力图完整呈现近现代世界和中国物理科学发展的全貌,是一部目前国内为数不多的兼具学术价值和阅读乐趣的经典物理丛书。

“中外物理学精品书系”另一个突出特点是，在把西方物理的精华要义“请进来”的同时，也将我国近现代物理的优秀成果“送出去”。物理学科在世界范围内的重要性不言而喻，引进和翻译世界物理的经典著作和前沿动态，可以满足当前国内物理教学和科研工作的迫切需求。另一方面，改革开放几十年来，我国的物理学研究取得了长足发展，一大批具有较高学术价值的著作相继问世。这套丛书首次将一些中国物理学者的优秀论著以英文版的形式直接推向国际相关研究的主流领域，使世界对中国物理学的过去和现状有更多的深入了解，不仅充分展示出中国物理学研究和积累的“硬实力”，也向世界主动传播我国科技文化领域不断创新的“软实力”，对全面提升中国科学、教育和文化领域的国际形象起到重要的促进作用。

值得一提的是，“中外物理学精品书系”还对中国近现代物理学科的经典著作进行了全面收录。20 世纪以来，中国物理界诞生了很多经典作品，但当时大都分散出版，如今很多代表性的作品已经淹没在浩瀚的图书海洋中，读者们对这些论著也都是“只闻其声，未见其真”。该书系的编者们在这方面下了很大工夫，对中国物理学科不同时期、不同分支的经典著作进行了系统的整理和收录。这项工作具有非常重要的学术意义和社会价值，不仅可以很好地保护和传承我国物理学的经典文献，充分发挥其应有的传世育人的作用，更能使广大物理学人和青年学子切身体会我国物理学研究的发展脉络和优良传统，真正领悟到老一辈科学家严谨求实、追求卓越、博大精深的治学之美。

温家宝总理在 2006 年中国科学技术大会上指出，“加强基础研究是提升国家创新能力、积累智力资本的重要途径，是我国跻身世界科技强国的必要条件”。中国的发展在于创新，而基础研究正是一切创新的根本和源泉。我相信，这套“中外物理学精品书系”的出版，不仅可以使所有热爱和研究物理学的人们从中获取思维的启迪、智力的挑战和阅读的乐趣，也将进一步推动其他相关基础科学更好更快地发展，为我国今后的科技创新和社会进步做出应有的贡献。

“中外物理学精品书系”编委会 主任

中国科学院院士，北京大学教授

王恩哥

2010 年 5 月于燕园

Preface

Throughout the history of civilization, mathematics has served as one of the major tools to analyze real world applications. In turn, through these applications it has been developed and expanded considerably. Moreover, mathematics helps establishing and consolidating the belief in eternal and exact truth, and hence the trust on sciences.

Since the invention of Calculus by Newton and Leibniz in the seventeenth century, mathematics has been overwhelmingly successful in almost every branch of sciences. It is instrumental for scientists and engineers to think, to work and to communicate.

We recall that Calculus is built fundamentally upon the definition of the real numbers. This definition naturally leads to the notion of limit. Two special and most useful limits are the derivative and the integral of a function. Most physical theories are described in terms of differential equations. Modern physics essentially started from Newton's theory of motions, and Newton's second law is a paradigm. The electro-magnetic theory is essentially the studies on the Maxwell equations. The theory of general relativity explores the Einstein equation, and the quantum mechanics uses the Schrödinger equation or the Wigner equation. We discuss the Lagrangian or Hamiltonian systems in mechanics, the biharmonic equation in elasticity, and the Navier-Stokes equations in fluid, etc.

The invention of electronic computers changed fundamentally the way for scientific research. Though we may not obtain analytical solution to a complex system in general, computer allows us to find the solution to a set of continuous differential equations in a discrete manner. Under certain circumstances, we even do not need to go to the continuous form. For instance, a fully discrete binomial algorithm may be used to compute the price of an option. We point out that instead of becoming a substitute for the continuous analysis, scientific computing reaches its best efficiency in real world applications only when we have a good understanding of the physics, the continuous modeling and analysis, the numerical algorithm, and the computer code.

This book is an outcome of an advanced course, conducted in English, for graduate students and senior undergraduate students at Department of Mechanics of Peking University. This course has been offered roughly every other year since 1998. We set forth the following objectives.

- To show some modern (1900-1990?) mathematical methods that are widely used in engineering sciences, nonlinear mechanics and other physical sciences.
- To help initiating research activities, namely, to boost ideas, to formulate the problem, and to explore the mathematics.
- To help bridging the gap between the mathematical tools and the physical understandings taught in other undergraduate courses.

A major ingredient of this course is nonlinearity. As is well known, superposition is the feature that distinguishes linear and nonlinear systems.

In linear algebra, we have

$$Ax_1 = y_1, \quad Ax_2 = y_2 \quad \Rightarrow \quad A(x_1 + x_2) = y_1 + y_2.$$

The differential operator and integral operator are also linear.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\alpha f(x) + \beta g(x)) = \alpha \frac{\mathrm{d}f}{\mathrm{d}x} + \beta \frac{\mathrm{d}g}{\mathrm{d}x},$$
$$\int (\alpha f(x) + \beta g(x))\, \mathrm{d}x = \alpha \int f(x)\mathrm{d}x + \beta \int g(x)\mathrm{d}x.$$

Similarly, the Fourier transform and the Laplace transform are linear operators.

Superposition also applies to linear differential equations. For example, if both $x_1(t)$ and $x_2(t)$ are solutions to the equation $ax'' + bx' + cx = 0$, so is $\alpha x_1(t) + \beta x_2(t)$ for any constant α and β. As a matter of fact, the general solution is $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, where λ_1 and λ_2 are the roots to the quadratic equation $a\lambda^2 + b\lambda + c = 0$.

For instance, we consider the following RCL circuit in Fig. 1. A resistor obeys Ohm's law $V_R = RI$, while an inductor and a capacitor satisfy $V_L = L\dfrac{\mathrm{d}I}{\mathrm{d}t}$ and $I =$

$C\frac{\mathrm{d}V_C}{\mathrm{d}t}$, respectively. Kirchhoff's law gives rise to an integral-differential equation.

$$V = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}\int_0^t I(s)\mathrm{d}s + V_C(0).$$

We differentiate it once to obtain

$$L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = 0.$$

If V varies along with time, the righthand side does not vanish. This circuit may generate electro-magnetic waves of a certain frequency.

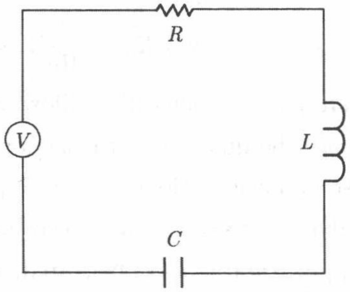

Figure 1 RCL circuit.

As Einstein pointed out, the Laws of Nature cannot be linear. Linear system is usually a special case or a simplified version, therefore incomplete. Lots of important features of the real world can only be explained under the framework of nonlinear systems. Besides, linear problems are relatively simple, and mathematical tools we have learned before are fairly competent to handle them. We head for challenges and excitements through studies of nonlinear problems.

For an example of nonlinear system, we consider an oversimplified mechanical system consisted of the sun with mass M and the earth with mass m in Fig. 2. Let their positions be y and x, respectively. Newton's second law and the universal gravitation theory lead to the following coupled system,

$$\begin{cases} m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{GmM}{|x-y|^2}, \\ M\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{GmM}{|x-y|^2}. \end{cases}$$

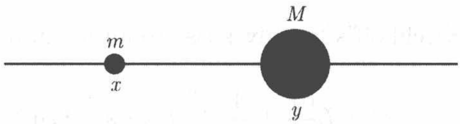

Figure 2 A one-dimensional two-body system of the sun and the earth.

For another example, the Navier-Stokes equations for an incompressible fluid are as follows. Let $\mathbf{v}$ be the velocity, p the pressure, and Re the Reynolds number,

$$\begin{cases} \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}_t + (\mathbf{v}\nabla) \cdot \mathbf{v} + \nabla p = \dfrac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}. \end{cases}$$

In this course, we shall mainly discuss the following topics.

In Chapter 1, we expose the qualitative theory for ODE systems (4 weeks of teaching). We start with some basic notions. Then we present a basic fixed point theory from functional analysis. This allows us to establish existence results for an ODE system. A further application is also illustrated, namely, iteration methods for solving a linear algebraic system. To understand qualitatively an ODE system, we analyze its critical points. For a second order ODE, the so-called plane analysis may provide substantial understanding. For a general system, there are not as many powerful tools. Stability analysis via the Lyapunov function is an exception. When there is a controlling parameter in a system, bifurcation may occur. We conclude this chapter by an exhibition of chaos in the Lorenz system and the logistic map.

For partial differential equations, we first study reaction-diffusion systems (3 weeks of teaching). We set up BVP (boundary-value problem) and IBVP (initial-boundary-value problem), and then show a simple example of instability at equilibrium. For a linearized problem, its dispersion relation gives a primary linear stability result. For nonlinear systems, an invariant domain approach sometimes works. This is a geometrical way to get *a priori* estimate. For a special example of nonlinear system, we illustrate a perturbation method for its steady states. Next, traveling wave analysis reduces a PDE system to an ODE system, and usually provides explanation to some wave behaviours of the PDE system. Only for very exceptional cases, a nonlinear PDE may be transformed to a linear one, e.g., Burgers' equation by the Cole-Hopf transform. We further illustrate a combination of theoretical and numerical investigations in an example of reaction-diffusion equation, namely, the evolutionary Duffing equation.

In Chapter 3, we discuss elliptic equations (2 weeks of teaching). the main topic is to introduce some basic ideas in the modern theories of partial differential equations. We start with generalized functions and weak derivatives, and introduce briefly the Sobolev spaces, and state the embedding theorem. Weak formulations and minimization procedure are used to establish existence results.

Chapter 4 is devoted to hyperbolic conservation laws (5 weeks of teaching). The most distinct feature of this type of PDE's lies in the inevitable appearance of discontinuities, regardless of smooth initial data. We show shock formation in inviscid Burgers' equation, by a characteristics approach. Then taking the Euler equations for polytropic gas as an example, we discuss the elementary waves, which include shock waves via vanishing viscosity approach, and rarefaction waves via self-similarity solution approach. For a general Riemann problem of gas dynamics, the unique composition of these elementary waves gives the solution, which is a weak one by construction. We further discuss solitons in the KdV equation, for which a brilliant theory of inverse scattering transform is sketched.

As this book is only an introduction of qualitative theories for ODE and PDE systems, further readings are suggested.

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Contents

Chapter 1 Qualitative Theory for ODE Systems

1.1 Basic notions

For many applications, we describe a system using only one independent variable. A dependent quantity is expressed as a function of this independent variable. An ordinary differential equation (ODE) is an equation that contains an unknown function, called a state variable, together with its derivatives with respect to the single independent variable. For historical reasons, the independent variable is typically denoted as t, representing time. Depending on the applications, actually t may mean some other quantities, such as temperature, height, etc. The order of the highest derivative in an ODE is its order.

An ODE system, also called a dynamical system, is a set of ODE's. Typically each equation in this system is of first order. The order of the ODE system is the number of first order equations in the system.

A high-order ODE can always be recast to an ODE system of the same order. For instance,

$$x'' + xx' + x(1-x) + f(t) = 0 \tag{1.1}$$

can be rewritten as

$$\begin{cases} x' = y, \\ y' = -[xy + x(1-x) + f(t)]. \end{cases} \tag{1.2}$$

Therefore, a general ODE system reads

$$x' = f(t, x), \quad \text{with} \quad x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n. \tag{1.3}$$

The ODE system is autonomous if the righthand side depends only on the state variable, that is, $f(x,t) = f(x)$. A non-autonomous system can be trivially reshaped to an autonomous one. In fact, if we take $y = (t, x_1, \cdots, x_n)^T$, then we obtain

$$y' = \begin{pmatrix} 1 \\ f(y) \end{pmatrix}. \tag{1.4}$$

Through this procedure, the order of the system rises by one.

In this course, we are not concerned with a particular solution to an ODE system. Instead, we take a global view of all the solutions to a system, and for the aforementioned reason, an autonomous ODE system. These solutions form a family of (vector) functions. This family is the object for the qualitative theory.

If one such function $x(t)$ satisfies $x(t_0) = x_0$ for a certain time t_0, then we call it an orbit, or a trajectory passing through the point (t_0, x_0). These names reflect that we take a geometrical view. Sometimes we use the notation $x = \phi(t; t_0, x_0)$ to identify the orbit. In contrast, in the previous ODE course, one regards (t_0, x_0) as an initial point, and usually considers the solution $x = x(t)$ only for $t \geqslant t_0$. The geometrical name for this part of the solution is the positive semi-orbit, denoted by $x = \phi^+(t; t_0, x_0)$. Meanwhile, the solution for $t \in (-\infty, t_0]$ is called as the negative semi-orbit, and denoted as $x = \phi^-(t; t_0, x_0)$. Under such a geometrical view, we regard the function $x(t)$ equivalent to a curve in the space $\mathbb{R}^n$. This space is called a phase space, or a phase plane if $n = 2$.

We remark that sometimes one also specifies boundary data, namely n algebraic equations for n quantities selected from $x_1(a), \cdots, x_n(a), x_1(b), \cdots, x_n(b)$, when one looks for a solution in $t \in [a, b]$.

For an autonomous system, a translation in time is invariant. More precisely, if $x = \phi(t; 0, x_0)$ solves the system with initial data $x(0) = x_0$, then $x = \phi(t+t_0; t_0, x_0)$ solves the system with initial data $x(t_0) = x_0$. Therefore, it suffices to study the problem with initial data at one selected time, which is usually chosen as $t_0 = 0$.

We recall that the existence, uniqueness and continuous dependency hold for quite general cases, e.g., when the source term (righthand side) is continuous. Existence will be discussed later by means of a fixed point theory.

Qualitative theory is concerned with the global structure of trajectories in the phase space, instead of a particular solution for certain given initial data.

At each given point x, the source term introduces a vector $f(x)$ in the phase space. The direction of this vector determines the direction of the trajectory, and the absolute value determines how fast a solution takes to go through this point. We may imagine that there is a particle moving along the trajectory according to the vector field (velocity field). See Fig. 1.1.

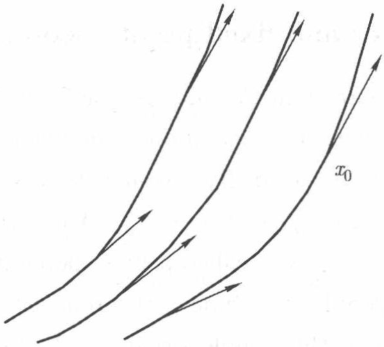

Figure 1.1 Trajectories, vector field and phase space (plane).

We notice that at a point x where $f(x)$ vanishes, the previous statement becomes meaningless. This leads to the notion of a critical point (equilibrium point, singular point, stationary point, etc.), which turns out to be crucial in later discussions.

A point x is a critical point where $f(x) = 0$. It is a regular point if f is finite and non-zero. Two trajectories may intersect only at a critical point in the phase space. This can be proved by the uniqueness of solution to the following ODE system in the neighborhood of a regular point x^*. Assuming that $f_1(x) \neq 0$, we have

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x)}{f_1(x)}, \cdots, \frac{\mathrm{d}x_n}{\mathrm{d}x_1} = \frac{f_n(x)}{f_1(x)}, \quad (x_2(x_1^*), \cdots, x_n(x_1^*) = (x_2^*, \cdots, x_n^*)). \tag{1.5}$$

Furthermore, a trajectory usually starts from/ends at a critical point or infinity, or forms a closed orbit. Under certain circumstances, a chaotic orbit may appear.

In the subsequent sections, we shall discuss the local existence for ODE systems, using a fixed point theory. Then we shall perform detailed analysis in the vicinity of a critical point.

1.2 Local existence

The local existence for an ODE system may be proven through a Picard iteration procedure. We present here a systematic approach, using a fixed point theory from functional analysis.