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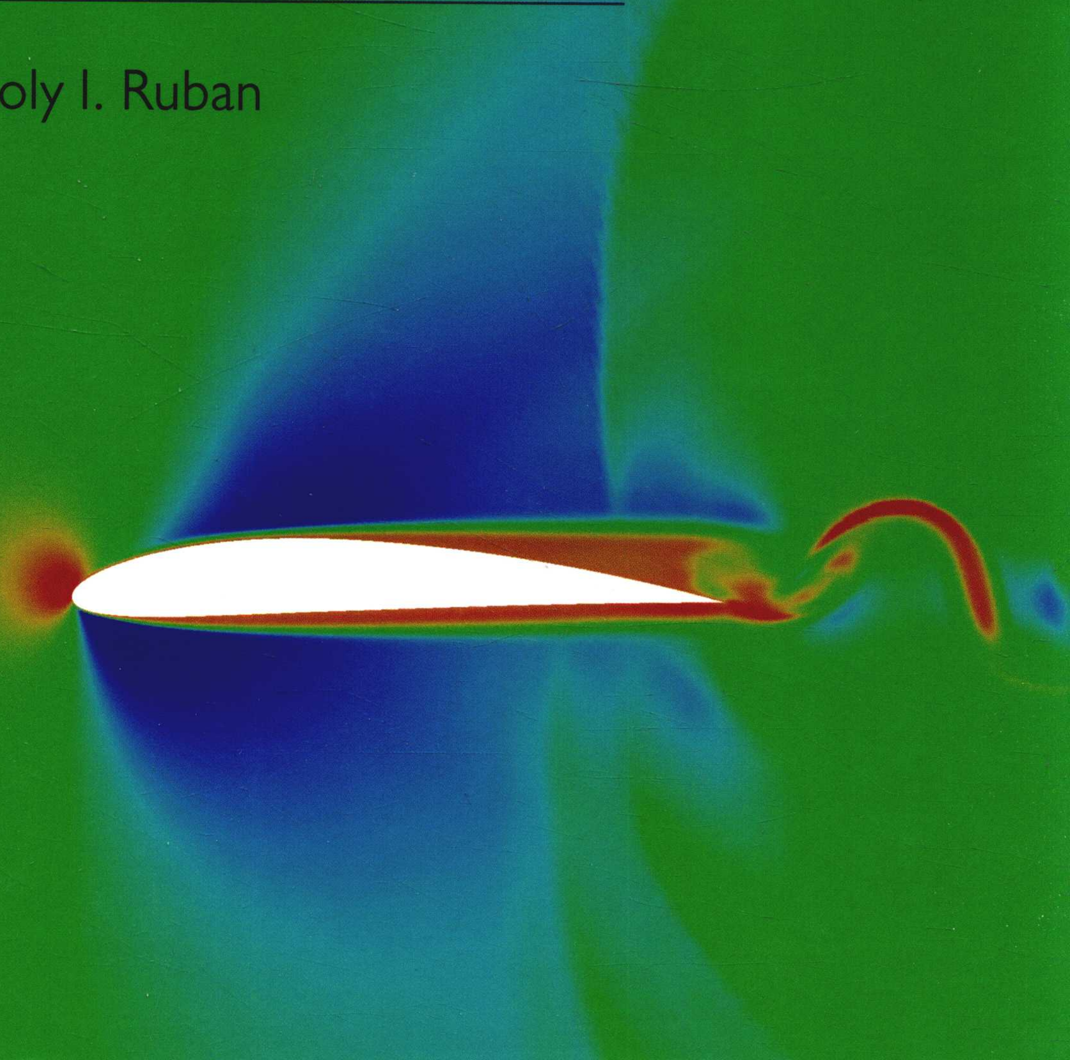
PART 2

Fluid Dynamics

ASYMPTOTIC PROBLEMS OF FLUID DYNAMICS

Anatoly I. Ruban



Fluid Dynamics

Part 2: Asymptotic Problems of Fluid Dynamics

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Fluid Dynamics

Preface

This is Part 2 of a book series on fluid dynamics that will comprise the following four parts:

Part 1. Classical Fluid Dynamics

Part 2. Asymptotic Problems of Fluid Dynamics

Part 3. Boundary Layers

Part 4. Hydrodynamic Stability Theory

The series is designed to give a comprehensive and coherent description of fluid dynamics, starting with chapters on classical theory suitable for an introductory undergraduate lecture course, and then progressing through more advanced material up to the level of modern research in the field. Our main attention is on high-Reynolds-number flows, both incompressible and compressible. Correspondingly, the target reader groups are undergraduate and MSc students reading mathematics, aeronautical engineering, or physics, as well as PhD students and established researchers working in the field.

In Part 1, we started with discussion of fundamental concepts of fluid dynamics, based on the *continuum hypothesis*. We then analysed the forces acting inside a fluid, and deduced the Navier–Stokes equations for incompressible and compressible fluids in Cartesian and curvilinear coordinates. These were employed to study the properties of a number of flows that are represented by the so-called *exact solutions* of the Navier–Stokes equations. This was followed by detailed discussion of the theory of inviscid flows for incompressible and compressible fluids. When dealing with incompressible inviscid flows, particular attention was paid to two-dimensional potential flows. These can be described in terms of the *complex potential*, allowing for the full power of the theory of functions of complex variable to be employed. We demonstrated how the method of conformal mapping might be used to study various flows of interest, such as flows past *Joukovskii aerofoils* and separated flows. For the later the *Kirchhoff model* was adopted. The final chapter of Part 1 was devoted to compressible flows of a perfect gas, including supersonic flows. Particular attention was given to the theory of characteristics, which was used, for example, to analyse the *Prandtl–Meyer flow* over a body surface bend or a corner. The properties of shock waves were also discussed in detail for steady and unsteady flows.

In the present Part 2, we introduce the reader to *asymptotic methods*. Also termed the *perturbation methods*, they are now an inherent part of applied mathematics, and are used in different branches of physics, including fluid dynamics. Asymptotic methods played an important role in the progress achieved in fluid dynamics in the last century. In Chapter 1 of Part 2 we discuss the mathematical aspects of the asymptotic theory. We start with basic definitions, using for this purpose so-called *coordinate asymptotic expansions*. The properties of asymptotic expansions are illustrated by asymptotic analysis of integrals. This includes the discussion of the *Watson lemma* and of the method of *steepest descent*. However, our main attention is with *parametric asymptotic expansions*. We discuss in detail the *method of matched asymptotic expansions*, the *method of multiple scales*, the *method of strained coordinates*, and the *WKB method*.

Then in Chapters 2–5 we use the asymptotic approach to study various aspects of the inviscid flow theory. We start in Chapter 2 with a discussion of the *thin aerofoil theory* for subsonic flows. In addition to steady attached flows past thin aerofoils, we examine the unsteady flows and flows with separation. Then in Chapter 3 we turn our attention to supersonic flows past thin aerofoils. We first analyse the linear approximation to the governing Euler equations, which leads to a remarkably simple relationship between the slope of the aerofoil surface and the pressure, known as the *Ackeret formula*. We then extend our analysis to the second-order *Busemann approximation*. Chapter 3 concludes with the study of a rather slow process of attenuation of the perturbations in the far field, and formation of the *N-wave*.

Chapter 4 is devoted to *transonic flows*. These are the flows with the free-stream Mach number, M_∞, close to the unity, which is characteristic of a passenger aircraft cruise flight. We first consider the far-field behaviour in the two-dimensional flow past an arbitrary body, assuming that $M_\infty = 1$. It appears that the corresponding solution of the Euler equations can be found analytically in a self-similar form. We then turn our attention to transonic flows past thin aerofoils. In this case the Euler equations can be reduced to the *Kármán–Guderley equation*. The latter is nonlinear and difficult to solve analytically, but it turns into the Tricomi equation if considered in the *hodograph plane*. We discuss two exact solutions of this equation—the first describes the transonic flow separation at a corner of a rigid body contour, and the second the flow accelerating into the Prandtl–Meyer expansion fan.

In Chapter 5 we discuss the properties of inviscid *hypersonic flows*, that is flows with large values of the free-stream Mach number. We first assume that the body placed in the flow has a blunt nose. In this case the shape of the shock, forming in front of the body, and the entire flow between the shock and the body surface become independent of the Mach number, M_∞, provided that M_∞ is large enough. This result is known as the *hypersonic stabilization principle*. The *Newton–Busemann theory* is discussed next. Then we turn our attention to the flows past thin bodies. These flows can be studied using the so-called *unsteady flow analogy*. In particular, the effect of a rounded nose on the hypersonic flow past a thin body may be described using analogy with *blast waves*.

In the concluding Chapter 6 we turn our attention to viscous flow. The discussion of various aspects of viscous flow theory will continue in Parts 3 and 4 of this book series. Here, in Chapter 6 our interest is in the low-Reynolds-number flows. We consider two classical problems of the low-Reynolds-number flow theory: the flow past a sphere and the flow past a circular cylinder. In both cases the flow analysis leads to a difficulty, known as *Stokes paradox*. We shall show that this paradox can be resolved using the formalism of matched asymptotic expansions.

The material presented in this book is based on lectures given by the author at the Moscow Institute of Physics and Technology, the University of Manchester, and Imperial College London.

Contents

Introduction

During the last century there was a remarkable progress in fluid dynamics. It was facilitated by the development of *perturbation methods*. In fact, many important concepts of modern fluid dynamics, such as *transonic flows* or *boundary layers*, can only be properly defined using a suitable perturbation approach.

Of course, fluid dynamics is not the only branch of physics where a degree of idealization is used to formulate the governing equations. Neglecting 'small effects' is a natural way of developing physical theories. Perturbation methods provide formal mathematical foundation for this approach. An early example, where a study of a physical process led to formulation of a perturbation theory, was the celebrated work of Lagrange (1811, 1815) and Laplace (1799–1825) on celestial mechanics. It is well known that the mass of Sun is much larger than the mass of the planets in the Solar System. Therefore, when calculating the Earth's orbit, one can disregard (in the leading-order approximation) the existence of other planets. If a more accurate prediction is required, then, in the first instance, one has to take into account the influence of the Moon's gravitation. The correspondent 'perturbations' to the leading order solution are obtained using the mass ratio of the Moon and the Earth, $\varepsilon = m_{\text{Moon}}/m_{\text{Earth}}$, as a small parameter. The main challenge is to ensure that the resulting solution remains accurate for a long period of time. We shall discuss how this is done in Sections 1.5 and 1.6. Interestingly enough, the perturbation theory of Lagrange and Laplace played an instrumental role in the discovery of planet Neptune. Its existence was theoretically predicted in 1846 by L. C. Adams and U. Le Verrier based on the observed deviations in motion of the planet Uranus. An early account of their works was given by Airy (1947) who held the post of British Astronomer Royal at the time.

Simultaneously another important development took place. In 1843 Cauchy published a note concerning the well-known series of Stirling for the logarithm of the Euler's Gamma function

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{n=1}^{N} \frac{B_{2n}}{2n\,(2n-1)}\,\frac{1}{x^{2n-1}} + \dots, \tag{I.1}$$

where B_{2n} are the Bernoulli numbers. Cauchy pointed out that the series on the right hand side of this formula, despite being divergent for all values of x, may be used in computing $\ln\Gamma(x)$ when x is large and positive. In fact, it was shown that, having fixed the number N of terms taken, the absolute error committed by stopping the summation at the Nth term is less than the absolute value of the next succeeding term, and hence becomes arbitrarily small with increasing x.

The subject reappeared again after forty years in the investigation performed by Poincaré (1886) upon the irregular behaviour of solutions to linear ordinary differential equations of a certain type. Poincaré demonstrated that at large values of the

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argument x the solutions may be constructed formally in the form of series which are divergent but nevertheless represent an actual solution in the same way as the above formula for $\ln \Gamma(x)$. Poincaré applied to such series the name of *asymptotic expansions*. In Sections 1.1 and 1.3 we will consider a number of examples of this kind. In particular, we will show that the Airy function, $Ai(x)$, which is a solution of the Airy equation

$$\frac{d^2w}{dx^2} - x\,w = 0,$$

and may be represented at large values of the argument x as

$$Ai(x) = \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-\frac{2}{3}x^{3/2}}\left(1 - \frac{5}{48}x^{-3/2} + \cdots\right) \quad \text{as} \quad x \to \infty. \tag{I.2}$$

Formulae like (I.1) and (I.2) are termed *coordinate expansions*. These are asymptotic expansions with the independent variable x playing the role of a large or small parameter. We will discuss these in Sections 1.1 and 1.2. Then in Sections 1.3–1.7 we will turn our attention to the *parametric expansions*. The latter became a major tool of theoretical analysis in fluid dynamics. The motion of fluids is described by the Navier–Stokes equations.[1] These are nonlinear (more precisely, quasi-linear) partial differential equations, which, if solved, might be used to analyse a wide variety of complicated physical phenomena, including hydrodynamic instability and transition to turbulence, boundary-layer separation from a rigid body surface and formation of eddy wake, non-uniqueness and hysteresis of fluid flows, and so on. Because of the complexity of the processes involved, it is hardly surprising that direct analytical solution of the Navier–Stokes equations is impossible, except in a few rather simple situations.[2] That is why the theoretical analysis of fluid flows was always based on seeking possible simplifications that might be introduced in the Navier–Stokes equations when, say, Reynolds number Re is large and the flow may be treated as predominantly inviscid, or Mach number M_∞ is small and the flow behaves as if it were incompressible.

The parameters used in the asymptotic theory of fluid flows may be subdivided into two categories. To the first one belong the so-called *dynamic parameters*, such as the Reynolds number and Mach number, which explicitly appear in the non-dimensional form of the Navier–Stokes equations.[3] They determine the relative significance of competing physical processes taking place in moving fluid. Therefore, assuming one or more parameters small or large, and applying asymptotic analysis, not only allows us to derive simplified equations of motion but, what is no less important, reveals the physical mechanisms of a fluid-dynamic phenomenon considered.

In the second category are *geometric parameters* such as the aspect ratio of an aircraft wing λ, and its relative thickness ε with respect to the chord. A wealth of knowledge in the wing aerodynamics has been produced based on the assumption of large aspect ratio, $\lambda \gg 1$, which allows us to treat the flow over the wing as

[1]For derivation of the Navier–Stokes equation the reader is referred to Section 1.7 in Part 1 of this book series.

[2]See Section 2.1 in Part 1.

[3]These are equations (1.7.37) on page 71 in Part 1.

quasi-two-dimensional. Another widely used approximation arises from the thin wing assumption, $\varepsilon \ll 1$. For two-dimensional flows, the corresponding theory is termed the *thin aerofoil theory*. We will discuss it in Chapters 2 and 3 for subsonic and supersonic flows respectively. It is followed by the discussion of transonic flows in Chapter 4, where we assume that $M_\infty - 1 \ll 1$, and hypersonic flows in Chapter 5; in the latter case we assume that $M_\infty \gg 1$. The theories presented in Chapters 2–5 belong to the class of *regular perturbations* when a single asymptotic representation of the solution is valid in the entire flow field. In the concluding Chapter 6 we turn to *singular perturbations*. We consider the low-Reynolds-number flow past a sphere and circular cylinder. In both cases, to describe the flow one needs to use the *method of matched asymptotic expansions*.

This method takes its origin from the seminal paper by Prandtl (1904) on large-Reynolds-number flows. Before Prandtl's study, it was generally believed that fluid flows with low viscosity may be described by the Euler equations of inviscid fluid motion; the latter follow from the Navier–Stokes equations by setting $Re = \infty$. Prandtl noticed that while in a large Reynolds number flow past a rigid body the Euler equations really hold in the bulk of the flow, the inviscid description appeared to be invalid near the body surface. In a thin boundary layer adjacent to the wall another set of equations, known as *Prandtl's boundary-layer equations*, should be used.

Prandtl's idea of subdividing the entire flow field into two separate regions where different asymptotic forms of the governing equations apply, underwent thorough discussion in 1950s. Amongst those involved were Friedrichs (1953, 1954), Kaplun (1954, 1957, 1967), Kaplun and Lagerstrom (1957), Lagerstrom and Cole (1955), Cole (1957, 1968), and Van Dyke (1956, 1964). As a result of their studies the approach became a formal mathematical technique termed the *method of inner and outer expansions* or, after Bretherton (1962), the *method of matched asymptotic expansions*. A description of this method is given in Section 1.4 in the present Part 2. We will then use it in Part 3, devoted to the boundary-layer theory, and in Part 4, where the hydrodynamic stability theory is discussed.

1
Perturbation Methods

1.1 Coordinate Asymptotic Expansions

To introduce the notion of asymptotic expansion, we will start with the Taylor expansion which may be used in both ways: as a conventional *series* and as an *asymptotic expansion*.

1.1.1 Taylor expansion

Let us suppose that function $f(x)$ is defined on the interval (a, b) of real variable x, with a graph of this function shown in Figure 1.1. Let us further suppose that $f(x)$ has $N+1$ continuous derivatives on (a, b), where $N = 0, 1, 2, \dots$. To construct the Taylor expansion for $f(x)$ we choose an arbitrary point $x \in (a, b)$ and write

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(\xi)\, d\xi, \tag{1.1.1}$$

where x_0 also belongs to the interval (a, b), and is referred to as the centre of the sought expansion.

If $N > 0$, then the integral on the right-hand side of (1.1.1) can be evaluated using the integration by parts:

$$\int u\, dv = uv - \int v\, du.$$

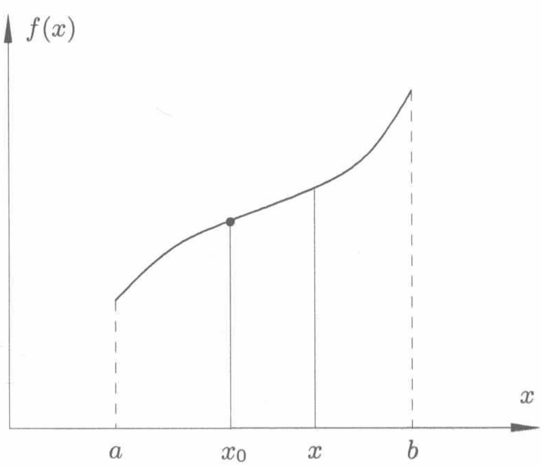

Fig. 1.1: Graph of function $f(x)$.

We choose

$$u = f'(\xi), \qquad dv = d\xi,$$
$$du = f''(\xi)\,d\xi, \qquad v = \xi - x,$$

and then (1.1.1) becomes

$$f(x) = f(x_0) + (\xi - x)f'(\xi)\Big|_{x_0}^{x} - \int_{x_0}^{x} (\xi - x)f''(\xi)\,d\xi$$
$$= f(x_0) + (x - x_0)f'(x_0) - \int_{x_0}^{x} (\xi - x)f''(\xi)\,d\xi.$$

Repeating this procedure N times, yields

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n + R_N(x), \tag{1.1.2}$$

where $R_N(x)$ is the *remainder term* given by

$$R_N(x) = \frac{1}{N!}\int_{x_0}^{x} (x - \xi)^N f^{(N+1)}(\xi)\,d\xi. \tag{1.1.3}$$

Our intention is to use equation (1.1.2) without the remainder term:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{1.1.4}$$

It is obvious that the sum on the right-hand side of (1.1.4), being called the *Taylor expansion* of function $f(x)$, will represent $f(x)$ properly if $R_N(x)$ is small. There are two ways to satisfy this requirement. First, if $f(x)$ has infinite number of derivatives, then we can increase the number of terms in (1.1.4) hoping that due to the coefficient $1/N!$ in front of the integral in (1.1.3) the remainder term will become small for a chosen point x, or a range of points from (a, b). Indeed, under certain conditions imposed upon $f(x)$, the remainder term $R_N(x)$ can be shown to tend to zero as $N \to \infty$, leading to the *Taylor series*:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{1.1.5}$$

It represents a particular example of *convergent series*. The latter is defined as follows.

Definition 1.1 *Series* (1.1.5) *is said to converge to* $f(x)$ *at a point* x *of the interval* (a, b) *if, given arbitrary* $\epsilon > 0$, *it is possible to find large enough* N_0 *such that*[1]

$$\left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right| < \epsilon$$

for all $N > N_0$.

The Taylor series are of particular interest in complex analysis. It is known that any function $f(z)$ of the complex variable z, which has at least one continuous derivative in an open region $\mathcal{D}$ of the complex plane, is infinitely differentiable in $\mathcal{D}$. It therefore may be represented by the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \tag{1.1.6}$$

where the centre z_0 should be positioned inside region $\mathcal{D}$. The series on the right-hand side of (1.1.6) is known to converge to $f(z)$ for all z within a circle $|z - z_0| < R$, where radius R equals the minimal distance from z_0 to a point in the complex plane where $f(z)$ fails to be analytic.

To illustrate this statement let us consider, as an example, the following function:

$$f(z) = \frac{1}{z+1}.$$

Notice that this function has a singularity at $z = -1$. Since

$$f^{(n)}(z) = (-1)^n \, n!(z+1)^{-(n+1)},$$

the Taylor series for $f(z)$, centred at $z_0 = 0$, is written as

$$\frac{1}{z+1} = \sum_{n=0}^{\infty} (-1)^n z^n. \tag{1.1.7}$$

The radius R of convergence of (1.1.7) is easily calculated using the *root test.*[2] It states that a series

$$f(z) = \sum_{n=0}^{\infty} w_n(z)$$

is convergent, if

$$M = \overline{\lim_{n\to\infty}} \, |w_n|^{1/n} < 1,$$

and is divergent, if $M > 1$. Since the terms of the series (1.1.7) are $w_n(z) = (-1)^n z^n$, it follows that $M = |z|$. Thus the convergence radius is $R = 1$, that is the series (1.1.7) converges for $|z| < 1$ and is divergent for $|z| > 1$, with the singular point $z = -1$ situated on the boundary of the convergence region; see Figure 1.2.

[1] In general N_0 depends on a position x inside the interval (a, b). If N_0 may be chosen independently of x then the series (1.1.5) is said to converge to $f(x)$ uniformly on (a, b).

[2] See, for example, Dettman (1965).

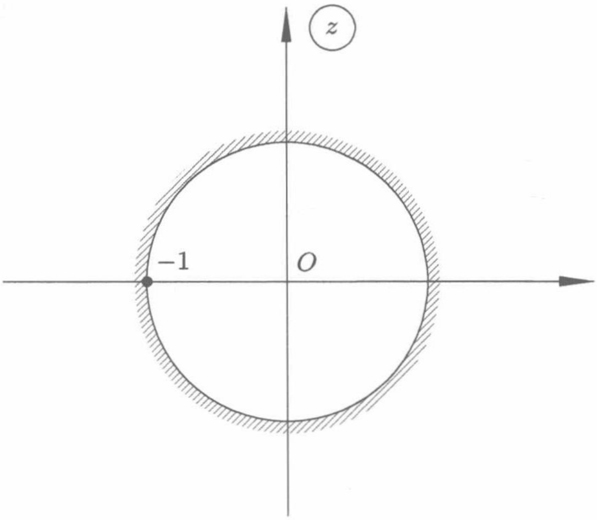

Fig. 1.2: The circle of convergence of the Taylor series (1.1.7).

Let us now return to equations (1.1.2), (1.1.3), and suppose that function $f(x)$ has only a finite number of derivatives or, for some reason, calculating higher derivatives proves difficult. The Taylor expansion can nevertheless be used to represent $f(x)$ provided that x lies close to x_0. Indeed the remainder term (1.1.3) may be bounded as

$$|R_N(x)| \leq \frac{1}{N!} \int_{x_0}^{x} |x-\xi|^N \left|f^{(N+1)}(\xi)\right| |d\xi|.$$

If $\left|f^{(N+1)}(\xi)\right|$ is bounded on (a,b), that is there exists a positive constant L such that

$$\left|f^{(N+1)}(\xi)\right| < L \quad \text{for all} \quad \xi \in [x_0, x],$$

then

$$|R_N(x)| \leq \frac{L}{N!} \int_{x_0}^{x} |x-\xi|^N d|\xi|,$$

which is easily integrated to yield[3]

$$|R_N(x)| \leq -\frac{L}{(N+1)!}|x-\xi|^{N+1}\Bigg|_{x_0}^{x} = \frac{L}{(N+1)!}|x-x_0|^{N+1}. \tag{1.1.8}$$

Hence, given an arbitrary $\epsilon > 0$, it is possible to find $\delta(\epsilon) > 0$, such that

$$|R_N(x)| = \left| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \right| < \epsilon$$

[3]The integration may be performed by assuming first that $x > x_0$, and then the calculations can be repeated for $x < x_0$.

for all x from the δ-vicinity of point x_0, namely, for all x satisfying the condition

$$|x - x_0| < \delta(\epsilon).$$

Treated in this way the Taylor expansion (1.1.4) represents an example of *asymptotic expansion.*

Using (1.1.4) with $x_0 = 0$, the Taylor expansions for the following elementary functions may be easily found to be

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{1.1.9a}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \tag{1.1.9b}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \tag{1.1.9c}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \tag{1.1.9d}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots. \tag{1.1.9e}$$

1.1.2 Asymptotic expansion of an integral

To give another example of an asymptotic expansion, let us consider the function

$$\Psi(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi. \tag{1.1.10}$$

Our task will be to evaluate $\Psi(x)$ at large values of x. To perform this task we rewrite the integral in (1.1.10) as

$$\int_x^\infty e^{-\xi^2}\, d\xi = \int_x^\infty \frac{e^{-\xi^2} 2\xi}{2\xi}\, d\xi,$$

and use the integration by parts with

$$u = \frac{1}{2\xi}, \qquad dv = e^{-\xi^2} 2\xi\, d\xi,$$
$$du = -\frac{1}{2\xi^2}\, d\xi, \qquad v = -e^{-\xi^2}.$$

We have

$$\int_x^\infty e^{-\xi^2}\, d\xi = -\frac{e^{-\xi^2}}{2\xi}\bigg|_x^\infty - \int_x^\infty \frac{e^{-\xi^2}}{2\xi^2}\, d\xi = \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-\xi^2}}{2\xi^2}\, d\xi.$$