高观点下的初等数学

算术 代数 分析

Elementary Mathematics from an Advanced Standpoint

Arithmetic, Algebra, Analysis

[德]F.C. 克莱因 著

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FELIX CHRISTIAN KLEIN

ELEMENTARY MATHEMATICS

FROM AN ADVANCED STANDPOINT

ARITHMETIC • ALGEBRA • ANALYSIS

TRANSLATED FROM THE THIRD GERMAN EDITION BY

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Preface to the First Edition.

The new volume which I herewith offer to the mathematical public, and especially to the teachers of mathematics in our secondary schools, is to be looked upon as a first continuation of the lectures *Über den mathematischen Unterricht an den höheren Schulen**, in particular, of those on *Die Organisation des mathematischen Unterrichts*** by *Schimmack* and me, which were published last year by Teubner. At that time our concern was with the different ways in which the problem of instruction can be presented to the mathematician. At present my concern is with developments in the subject matter of instruction. I shall endeavor to put before the teacher, as well as the maturing student, from the view-point of modern science, but in a manner as simple, stimulating, and convincing as possible, both the content and the foundations of the topics of instruction, with due regard for the current methods of teaching. I shall not follow a systematically ordered presentation, as do, for example, Weber and Wellstein, but I shall allow myself free excursions as the changing stimulus of surroundings may lead me to do in the course of the actual lectures.

The program thus indicated, which for the present is to be carried out only for the fields of *Arithmetic*, *Algebra*, and *Analysis*, was indicated in the preface to Klein-Schimmack (April 1907). I had hoped then that Mr. Schimmack, in spite of many obstacles, would still find the time to put my lectures into form suitable for printing. But I myself, in a way, prevented his doing this by continuously claiming his time for work in another direction upon pedagogical questions that interested us both. It soon became clear that the original plan could not be carried out, particularly if the work was to be finished in a short time, which seemed desirable if it was to have any real influence upon those problems of instruction which are just now in the foreground. As in previous years, then, I had recourse to the more convenient method of *lithographing* my lectures, especially since my present assistant, Dr. Ernst Hellinger, showed himself especially well qualified for this work. One should not underestimate the service which Dr. Hellinger rendered. For it is a far cry from the spoken word of the teacher, influenced as it is by accidental conditions, to the subsequently polished and readable record.

* On the teaching of mathematics in the secondary schools.

** The organization of mathematical instruction.

In precision of statement and in uniformity of explanations, the lecturer stops short of what we are accustomed to consider necessary for a printed publication.

I hesitate to commit myself to still further publications on the teaching of mathematics, at least for the field of *geometry*. I prefer to close with the wish that the present lithographed volume may prove useful by inducing many of the teachers of our higher schools to renewed use of independent thought in determining the best way of presenting the material of instruction. This book is designed solely as such a mental spur, not as a detailed handbook. The preparation of the latter I leave to those actively engaged in the schools. It is an error to assume, as some appear to have done, that my activity has ever had any other purpose. In particular, the *Lehrplan der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte** (the so-called "Meraner" *Lehrplan*) is not mine, but was prepared, merely with my cooperation, by distinguished representatives of school mathematics.

Finally, with regard to the method of presentation in what follows, it will suffice if I say that I have endeavored here, as always, to combine geometric intuition with the precision of arithmetic formulas, and that it has given me especial pleasure to follow the historical development of the various theories in order to understand the striking differences in methods of presentation which parallel each other in the instruction of today.

Göttingen, June, 1908

Klein.

Preface to the Third Edition.

After the firm of Julius Springer had completed so creditably the publication of my collected scientific works, it offerred, at the suggestion of Professor Courant, to bring out in book form those of my lecture courses which, from 1890 on, had appeared in lithographed form and which were out of print except for a small reserve stock.

These volumes, whose distribution had been taken over by Teubner, during the last decades were, in the main, the manuscript notes of my various assistants. It was clear to me, at the outset, that I could not undertake a new revision of them without again seeking the help of younger men. In fact I long ago expressed the belief that, beyond a certain age, one ought not to publish independently. One is still qualified, perhaps, to direct in general the preparation of an edition, but is not able to put the details into the proper order and to take into proper account recent advances in the literature. Consequently I accepted the

* Curriculum prepared by the commission on instruction of the Society of German Natural Scientists and Physicians.

offer of Springer only after I was assured that liberal help in this respect would be provided.

These lithographed volumes of lectures fall into two series. The older ones are of special lectures which I gave from time to time, and were prepared solely in order that the students of the following semester might have at hand the material which I had already treated and upon which I proposed to base further work. These are the volumes on *Non-Euclidean Geometry, Higher Geometry, Hypergeometric Functions, Linear Differential Equations, Riemann Surfaces,* and *Number Theory*. In contrast to these, I have published several lithographed volumes of lectures which were intended, from the first, for a larger circle of readers. These are:

a) The volume on *Applications of Differential and Integral Calculus to Geometry*, which was worked up from his manuscript notes by C. H. Müller. This was designed to bridge the gap between the needs of applied mathematics and the more recent investigations of pure mathematicians.

b) and c) Two volumes on *Elementary Mathematics from an Advanced Standpoint*, prepared from his manuscript notes by E. Hellinger. These two were to bring to the attention of secondary school teachers of mathematics and science the significance for their professional work of their academic studies, especially their studies in pure mathematics.

A thoroughgoing revision of the volumes of the second series seemed unnecessary. A smoothing out, in places, together with the addition of supplementary notes, was thought sufficient. With their publication therefore, the initial step is taken. Volumes b), c), a) (in this order) will appear as Parts I, II, III of a single publication bearing the title *Elementary Mathematics from an Advanced Standpoint*. The combining, in this way, of volume a) with volumes b) and c) will meet the approval of all who appreciate the growing significances of applied mathematics for modern school instruction.

Meantime the revision of the volumes of the first series has begun, starting with the volume on *Non-Euclidean Geometry*. But a more drastic recasting of the material will be necessary here if the book is to be a well-rounded presentation, and is to take account of the recent advances of science. So much as to the general plan. Now a few words as to the first part of the *Elementary Mathematics*.

I have reprinted the preface to the 1908 edition of b) because it shows most clearly how the volume came into existence[1]. The second edition (1911), also lithographed, contained no essential changes, and the minor notes which were appended to it are now incorporated into

[1] My co-worker, R. Schimmack, who is mentioned there, died in 1912 at the age of thirty-one years, from a heart attack with which he was seized suddenly, as he sat at his desk.

the text without special mention. The present edition retains[1], in the main, the text of the first edition, including such peculiarities as were incident to the time of its origin. Otherwise it would have been necessary to change the entire articulation, with a loss of homogeneity. But during the sixteen years which have elapsed since the first publication, science has advanced, and great changes have taken place in our school system, changes which are still in progress. This fact is provided for in the appendices which have been prepared, in collaboration with me, by Dr. Seyfarth (Studienrat at the local Oberrealschule). Dr. Seyfarth also made the necessary stylistic changes in the text, and has looked after the printing, including the illustrations, so that I feel sincerely grateful to him. My former co-workers, Messrs. Hellinger and Vermeil, as well as Mr. A. Walther of Göttingen, have made many useful suggestions during the proof reading. In particular, I am indebted to Messrs. Vermeil and Billig for preparing the list of names and the index. The publisher, Julius Springer has again given notable evidence of his readiness to print mathematical works in the face of great difficulties.

Göttingen, Easter, 1924

Klein.

Preface to the English Edition.

Professor *Felix Klein* was a distinguished investigator. But he was also an inspiring teacher. With the rareness of genius, he combined familiarity with all the fields of mathematics and the ability to perceive the mutual relations of these fields; and he made it his notable function, as a teacher, to acquaint his students with mathematics, not as isolated disciplines, but as an integrated living organism. He was profoundly interested in the teaching of mathematics in the secondary schools, both as to the material which should be taught, and as to the most fruitful way in which it should be presented. It was his custom, during many years, at the University of Göttingen, to give courses of lectures, prepared in the interest of teachers and prospective teachers of mathematics in German secondary schools. He endeavored to reduce the gap between the school and the university, to rouse the schools from the lethargy of tradition, to guide the school teaching into directions that would stimulate healthy growth; and also to influence university attitude and teaching toward a recognition of the normal function of the secondary school, to the end that mathematical education should be a continuous growth.

These lectures of Professor *Klein* took final form in three printed volumes, entitled *Elementary Mathematics from an Advanced Standpoint.*

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[1] New comments are placed in brackets.

They constitute an invaluable work, serviceable alike to the university teacher and to the teacher in the secondary school. There is, at present, nothing else comparable with them, either with respect to their skilfully integrated material, or to the fascinating way in which this material is discussed. This English volume is a translation of Part I of the above work. Its preparation is the result of a suggestion made by Professor *Courant*, of the University of Göttingen. It is the expression of a desire to serve the need, in English speaking countries, of actual and prospective teachers of mathematics; and it appears with the earnest hope that, in a rather free translation, something of the spirit of the original has been retained.

The Translators.

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Introduction

In recent years[1], a far reaching interest has arisen among university teachers of mathematics and natural science directed toward a suitable training of candidates for the higher teaching positions. This is really quite a new phenomenon. For a long time prior to its appearance, university men were concerned exclusively with their sciences, without giving a thought to the needs of the schools, without even caring to establish a connection with school mathematics. What was the result of this practice? The young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching.

There is now a movement to abolish this double discontinuity, helpful neither to the school nor to the university. On the one hand, there is an effort to impregnate the material which the schools teach with new ideas derived from modern developments of science and in accord with modern culture. We shall often have occasion to go into this. On the other hand, the attempt is made to take into account, in university instruction, the needs of the school teacher. And it is precisely in such comprehensive lectures as I am about to deliver to you that I see one of the most important ways of helping. I shall by no means address myself to beginners, but I shall take for granted that you are all acquainted with the main features of the chief fields of mathematics. I shall often talk of problems of algebra, of number theory, of function theory, etc., without being able to go into details. You must, therefore, be moderately familiar with these fields, in order to follow me. My task will always be to show you the *mutual connection between problems in*

[[1] Attention is again drawn to the fact that the wording of the text is, almost throughout, that of the lithographed volume of 1908 and that comments which refer to later years have been put into the appendices.]

the various fields, a thing which is not brought out sufficiently in the usual lecture course, and more especially to emphasize the relation of these problems to those of school mathematics. In this way I hope to make it easier for you to acquire that ability which I look upon as the real goal of your academic study: the ability to draw (in ample measure) from the great body of knowledge there put before you a living stimulus for your teaching.

Let me now put before you some documents of recent date which give evidence of widespread interest in the training of teachers and which contain valuable material for us. Above all I think here of the addresses given at the last *Meeting of Naturalists* held September 16, 1907, in Dresden, to which body we submitted the Proposals for the Scientific Training of Prospective Teachers of Mathematics and Science of the Committee on Instruction of the Society of German Naturalists and Physicians. You will find these Proposals as the last section in the Complete Report of this Committee[1] which, since 1904, has been considering the entire complex of questions concerning instruction in mathematics and natural science and has now ended its activity; I urge you to take notice, not only of these Proposals, but also of the other parts of this very interesting report. Shortly after the Dresden meeting there occurred a similar debate at the Meeting of German Philologists and Schoolmen in Basel, September 25, in which, to be sure, the mathematical-scientific reform movement was discussed only as a link in the chain of parallel movements occurring in philological circles. After a report by me concerning our aims in mathematical-natural science reform there were addresses by P. Wendland (Breslau) on questions in *Archeology*, Al. Brandl (Berlin) on *modern languages* and, finally, *Ad. Harnack* (Berlin) on *History and religion*. These four addresses appeared together in one broschure[2] to which I particulary refer you. I hope that this auspicious beginning will develop into further cooperation between our scientists and the philologists, since it will bring about friendly feeling and mutual understanding between two groups whose relations have been unsympathetic even if not hostile. Let us endeavor always to foster such good relations even if we do among ourselves occasionally drop a critical word about the philologists, just as they may about us. Bear in mind that you will later be called upon in the schools to work together with the philologists for the common good and that this requires mutual understanding and appreciation.

Along with this evidence of efforts which reach beyond the borders of our field, I should like to mention a few books which aim in the

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[1] *Die Tätigkeit der Unterrichtskommission der Gesellschaft deutscher Naturforscher und Ärzte*, edited by A. Gutzmer. Leipzig and Berlin, 1908.

[2] *Universität und Schule.* Addresses delivered by F. Klein, P. Wendland, Al. Brandl, Ad. Harnack. Leipzig 1907.

same direction in the mathematical field and which will therefore be important for these lectures. Three years ago I gave, for the first time, a course of lectures with a similar purpose. My assistant at that time, R. Schimmack, worked the material up and the first part has recently appeared in print[1]. In it are considered the different kinds of schools, including the university, the conduct of mathematical instruction in them, the interests that link them together, and other similar matters. In what follows I shall from time to time refer to things which appear there without repeating them. This makes it possible for me to extend somewhat those considerations. That volume concerns itself with the organization of school instruction. I shall now consider the mathematical content of the material which enters into that instruction. If I frequently advert to the actual conduct of instruction in the schools, my remarks will be based not merely upon indefinite pictures of how the thing might be done or even upon dim recollections of my own school days; for I am constantly in touch with Schimmack, who is now teaching in the Göttingen gymnasium and who keeps me informed as to the present state of instruction, which has, in fact, advanced substantially beyond what it was in earlier years. During this winter semester I shall discuss "the three great A's", that is arithmetic, algebra, and analysis, withholding geometry for a continuation of the course during the coming summer. Let me remind you that, in the language of the secondary schools, these three subjects are classed together as arithmetic, and that we shall often note deviations in the terminology of the schools as compared with that at the universities. You see, from this small illustration, that only living contact can bring about understanding.

As a second reference I shall mention the three volume *Enzyklopädie der Elementarmathematik* by H. Weber and J. Wellstein, the work which, among recent publications, most nearly accords with my own tendencies. For this semester, the first volume, *Enzyklopädie der elementaren Algebra und Analysis*, prepared by H. Weber[2], will be the most important. I shall indicate at once certain striking differences between this work and the plan of my lectures. In Weber-Wellstein, the entire structure of elementary mathematics is built up systematically and logically in the mature language of the advanced student. No account is taken of how these things actually may come up in school instruction. The presentation in the schools, however, should be psychological and not systematic. The teacher so to speak, must be a diplomat. He must take account of the psychic processes in the boy in order to grip his interest;

[1] Klein, F., *Vorträge über den mathematischen Unterricht an höheren Schulen.* Prepared by von R. Schimmack. Part 1. *Von der Organisation des mathematischen Unterrichts.* Leipzig 1907. This book is referred to later as "Klein-Schimmack".

[2] Second edition. Leipzig 1906. [Fourth edition, 1922, revised by P. Epstein. — Referred to as "Weber-Wellstein I".

and he will succeed only if he presents things in a form intuitively comprehensible. A more abstract presentation will be possible only in the upper classes. For example: The child cannot possibly understand if numbers are explained axiomatically as abstract things devoid of content, with which one can operate according to formal rules. On the contrary, he associates numbers with concrete images. They are numbers of nuts, apples, and other good things, and in the beginning they can be and should be put before him only in such tangible form. While this goes without saying, one should—mutatis mutandis—take it to heart, that in all instruction, even in the university, mathematics should be associated with everything that is seriously interesting to the pupil at that particular stage of his development and that can in any way be brought into relation with mathematics. It is just this which is back of the recent efforts to give prominence to applied mathematics at the university. This need has never been overlooked in the schools so much as it has at the university. It is just this psychological value which I shall try to emphasize especially in my lectures.

Another difference between Weber-Wellstein and myself has to do with defining the content of school mathematics. Weber and Wellstein are disposed to be conservative, while I am progressive. These things are thoroughly discussed in Klein-Schimmack. We, who are called the reformers, would put the function concept at the very center of instruction, because, of all the concepts of the mathematics of the past two centuries, this one plays the leading role wherever mathematical thought is used. We would introduce it into instruction as early as possible with constant use of the graphical method, the representation of functional relations in the xy system, which is used today as a matter of course in every practical application of mathematics. In order to make this innovation possible, we would abolish much of the traditional material of instruction, material which may in itself be interesting, but which is less essential from the standpoint of its significance in connection with modern culture. Strong development of space perception, above all, will always be a prime consideration. In its upper reaches, however, instruction should press far enough into the elements of infinitesimal calculus for the natural scientist or insurance specialist to get at school the tools which will be indispensable to him. As opposed to these comparatively recent ideas, Weber-Wellstein adheres essentially to the traditional limitations as to material. In these lectures I shall of course be a protagonist of the new conception.

My third reference will be to a very stimulating book: *Didaktik und Methodik des Rechnens und der Mathematik*[1] by Max Simon, who like

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[1] Second edition, Munich 1908. Separate reprint from Baumeister's *Handbuch der Erziehungs- und Unterrichtslehre für höhere Schulen*, first edition, 1895.

Weber and Wellstein is at Strassburg. Simon is often in agre ment with our views, but he sometimes takes the opposite standpoint; and inasmuch as he is a very subjective, temperamental, personality he often clothes these contrasting views in vivid words. To give one example, the proposals of the Committee on Instruction of the Natural Scientists require an hour of geometric propaedeutics in the second year of the gymnasium, whereas at the present time this usually begins in the third year. It has long been a matter of discussion which plan is the better; and the custom in the schools has often changed. But Simon declares the position taken by the Commission, which, mind you, is at worst open to argument, to be "worse than a crime", and that without in the least substantiating his judgment. One could find many passages of this sort. As a precursor of this book I might mention Simon's *Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis*[1].

After this brief introduction let us go over to the subject proper, which I shall consider under three headings, as above indicated.

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[1] Leipzig 1906.

First Part

Arithmetic

I. Calculating with Natural Numbers

We begin with the foundation of all arithmetic, calculation with positive integers. Here, as always in the course of these lectures, we first raise the question as to how these things are handled in the schools; then we shall proceed to the question as to what they imply when viewed from an advanced standpoint.

1. Introduction of Numbers in the Schools

I shall confine myself to brief suggestions. These will enable you to recall how you yourselves learned your numbers. In such an exposition it is, of course, not my purpose to induct you into the practice of teaching, as is done in the Seminars of the secondary schools. I shall merely exhibit the material upon which we shall base our critique.

The problem of teaching children the properties of integers and how to reckon with them, and of leading them on to complete mastery, is very difficult and requires the labor of several years, from the first school year until the child is ten or eleven years old. The manner of instruction as it is carried on in this field in Germany can perhaps best be designated by the words *intuitive* and *genetic*, i. e., the entire structure is gradually erected on the basis of familiar, concrete things, in marked contrast to the customary *logical* and *systematic* method at the university.

One can divide up this material of instruction roughly as follows: The entire first year is occupied with the integers from 1 to 20, the first half being devoted to the range 1 to 10. The integers appear at first as numbered pictures of points or as arrays of all sorts of objects familiar to the children. Addition and multiplication are then presented by intuitional methods, and are fixed in mind.

In the second stage, the integers from 1 to 100 are considered and the Arabic numerals, together with the notion of positional value and the decimal system, are introduced. Let us note, incidentally, that the name "Arabic numerals", like so many others in science, is a misnomer. This form of writing was invented by the Hindus, not by the Arabs. Another principal aim of the second stage is knowledge of the multiplication table. One must know what 5×7 or 3×8 is in one's sleep, so to speak. Consequently the pupil must learn the multiplication table by heart to this degree of thoroughness, to be sure only after it has been made clear to him visually with concrete things. To this end the *abacus* is used to advantage. It consists, as you all know, of 10 wires stretched one above another, upon each of which there are strung ten movable beads. By sliding these beads in the proper way, one can read off the result of multiplication and also its decimal form.

The third stage, finally, teaches calculation with numbers of more than one digit, based on the known simple rules whose general validity is evident, or should be evident, to the pupil. To be sure, this evidence does not always enable the pupil to make the rules completely his own; they are often instilled with the authoritative dictum: "It is thus and so, and if you don't know it yet, so much the worse for you!"

I should like to emphasize another point in this instruction which is usually neglected in university teaching. It is that the application of numbers to practical life is strongly emphasized. From the beginning, the pupil is dealing with numbers taken from real situations, with coins, measures, and weights; and the question, "*What does it cost?*", which is so important in daily life, forms the pivot of much of the material of instruction. This plan rises soon to the stage of problems, when deliberate thought is necessary in order to determine what calculation is demanded. It leads to the problems in proportion, alligation, etc. To the words *intuitive* and *genetic*, which we used above to designate the character of this instruction, we can add a third word, *applications*.

We might summarize the purpose of the number work by saying: *It aims at reliability in the use of the rules of operation, based on a parallel development of the intellectual abilities involved, and without special concern for logical relations.*

Incidentally, I should like to direct your attention to a contrast which often plays a mischievous role in the schools, viz., the contrast between the university-trained teachers and those who have attended normal schools for the preparation of elementary school teachers. The former displace the latter, as teachers of arithmetic, during or after the sixth school year, with the result that a regrettable discontinuity often manifests itself. The poor youngsters must suddenly make the acquaintance of new expressions, whereas the old ones are forbidden. A simple example is the different multiplication signs, the $\times$ being preferred by the elementary teacher, the point by the one who has attended the university. Such conflicts can be dispelled, if the more highly trained teacher will give more heed to his colleague and will try to meet him on common ground. That will become easier for you, if you will realize what high regard one must have for the performance of the elementary school teachers. Imagine what methodical training is ne-