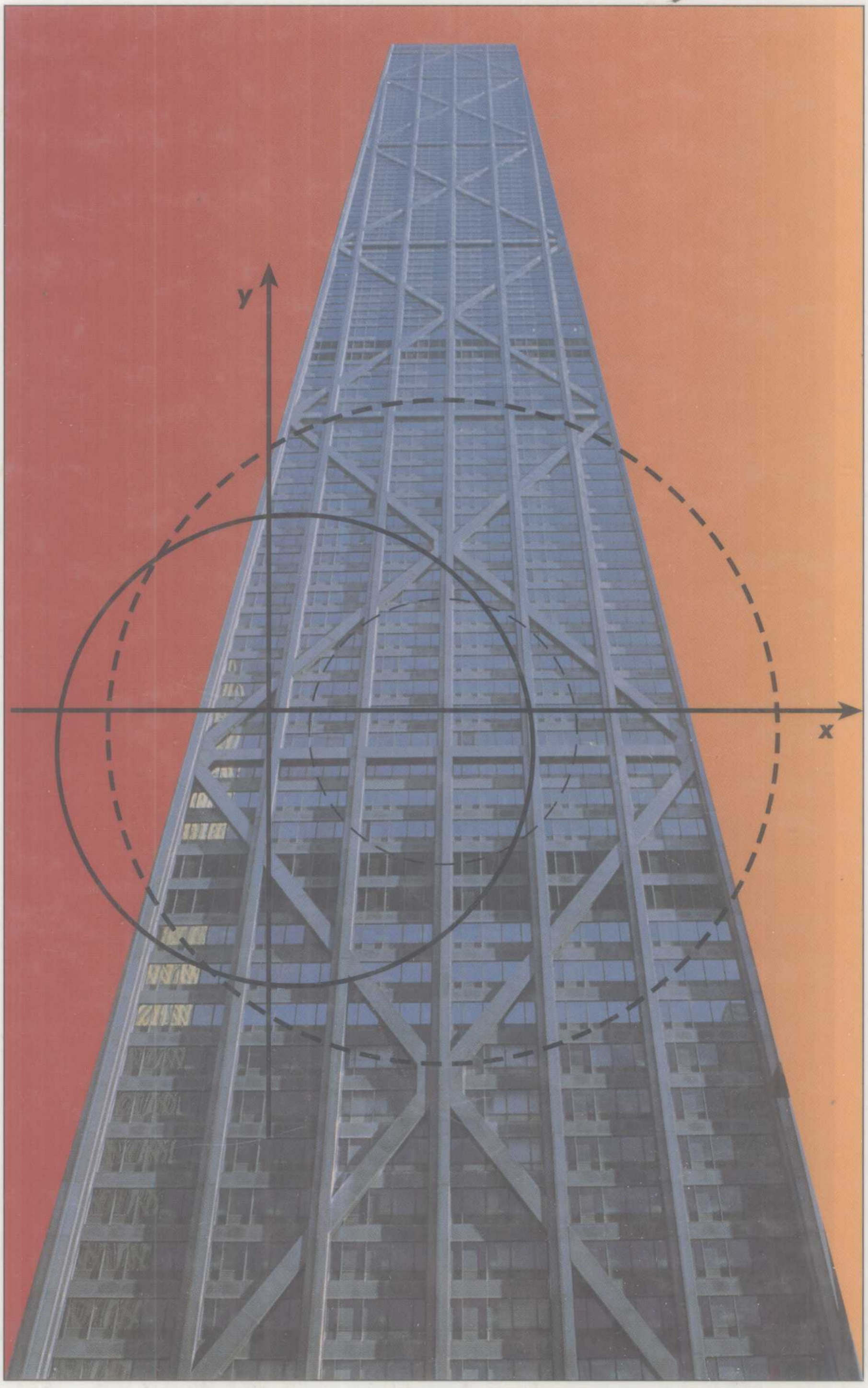
INTRODUCTION TO

ORDINARY DIFFERENTIAL EQUATIONS



STEPHEN H. SAPERSTONE

Introduction to Ordinary Differential Equations

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To my wife, Barbara, whose continual love, encouragement, and support have sustained me throughout this endeavor.

PREFACE

Goals

This book arose from the need to provide ample motivation, insight, and understanding in my introductory course in ordinary differential equations (ODEs) for engineering and science students. Far too often students get lost in the details of determining a solution formula to an ODE. Because the actual use of ODEs goes beyond such calculations, students cannot rely on the "plug 'n' chug" methods that adequately served most of them in calculus. Consequently, I have four primary goals:

1. To develop techniques for obtaining solutions for special types of ODEs (including the use of a computer algebra system such as *Maple*);
2. To squeeze as much information as possible from the ODE about its solutions without solving the ODE, even when a closed-form solution is possible;
3. To develop numerical approximation methods for solving initial value problems (IVPs), again even when a closed-form solution is possible; and
4. To illustrate by example how to model "real-world" phenomena with ODEs.

Approach

I employ a variety of approaches to enhance the reader's understanding of ODEs. Visualization tools, numerical estimation, symbolic computation, modeling, and applications are interspersed and integrated throughout the book. The applications not only add a sense of relevance to the study of ODEs, but also provide a common point of experience by which we can analyze and interpret their solutions. A great effort has been made to motivate most topics and to provide interpretations of a geometric or physical nature.

The style of this book is based in part on successful teaching strategies that I have used over many years. Typically, a new topic is introduced with a motivating example that demonstrates its need. In other words, I attempt to answer the question, "What is this good for?" Often I provide a "working definition" while properties of the topic are being explored. Another example is detailed with steps that suggest a generalization. When appropriate, I outline a solution procedure that is followed by yet another example. At this point, if appropriate, I introduce formal material. After a concluding example, an explanation, a justification, or even a proof is provided.

Features

- *Readability:* The language used to convey the material is more informal than most books. In the development of new concepts, lots of detail is provided in the examples.
- *Visualization tools:* Direction fields, level surfaces (for implicit solutions), and phase portraits are introduced as soon as possible and are used extensively throughout the book to interpret the properties of solutions.

- *Numerical tools:* Numerical methods are introduced early on. Euler's method for first-order ODEs is developed in Chapter 1 and is motivated by direction fields. Euler's method for second-order ODEs is introduced as soon as the concept of phase plane is introduced in Chapter 5. These methods are used to analyze many subsequent examples and applications where closed-form solutions are not possible or are too difficult to interpret.
- *Alerts:* Specific areas where students are more inclined to make a mistake are set off by an "ALERT."
- *Technology:* Where appropriate, subsections labeled "Technology Aids" illustrate how to use mathematical software. The examples include the complete *Maple* code for solving ODEs, displaying phase portraits, and computing numerical approximations. Although I have used *Maple* as the computer algebra system of choice, *Mathematica* or *Macsyma* will do just as well. *MATLAB* is also an excellent choice for numerical calculations and graphing. Finally, the author's *DIFF-E-Q* provides "quick and dirty" direction fields and graphs of solutions. Most sections have exercises identified by the icon 💻; this indicates the exercise should be done with software.
- *Applications:* Applications demonstrate the need for theory and illuminate the theory. Many of the applications are new for a book at this level.

- *Modeling:* Modeling is distinguished from applications in that modeling emphasizes how to create an ODE for a real-world problem. Modeling is introduced at length in Chapter 1 and pops up throughout the book. Identified by the icon shown in the margin at left, these discussions may be safely skipped by those readers and instructors who want to focus strictly on the mathematics.

Prerequisites and Audience Level

It is assumed that the reader has had the standard three-semester calculus sequence. Some familiarity with complex numbers and matrices would be nice, although appendices offer brief summaries of this basic material including power series. This book is written to reach a broad range of students. Enough detail is included in many of the procedures, examples, and applications so as to reach out to some of the less-prepared students. To those students with better than average preparation and ability, the book includes many advanced features, such as continuous dependence, a geometric interpretation of convolution, and a wide array of interesting examples (which are not normally included at this level).

Organization and Content

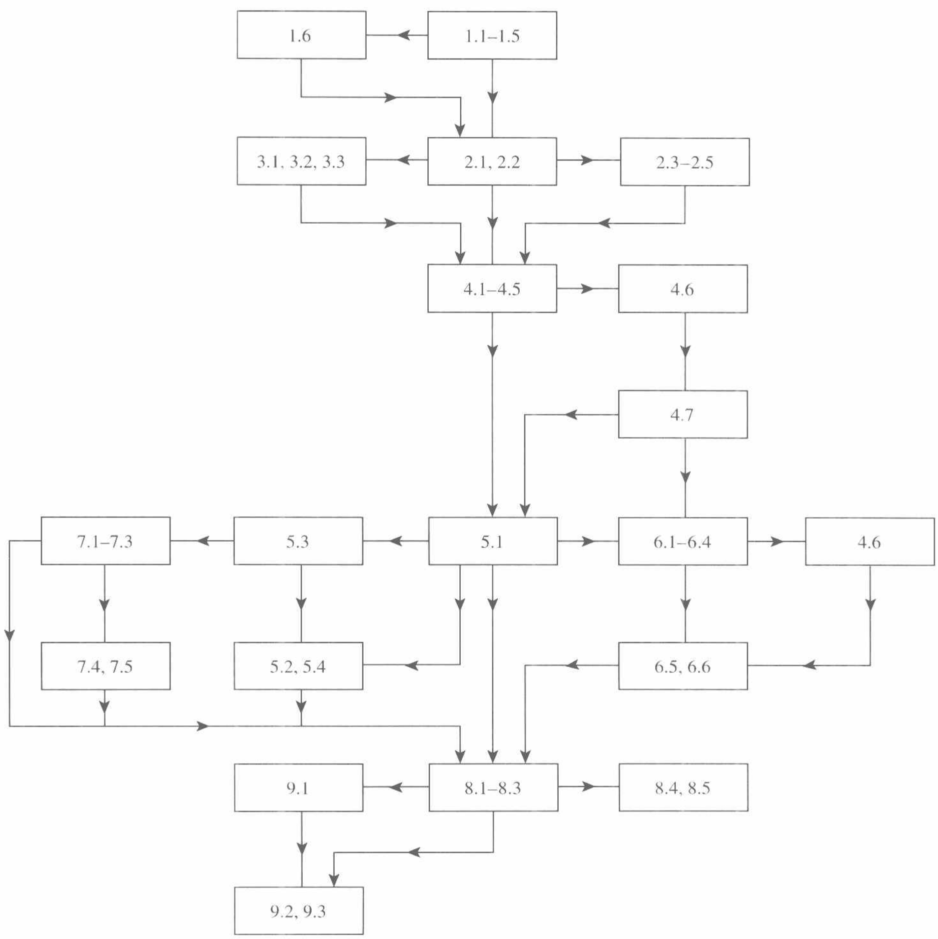
There are many paths through this book. In the table on the next page, I have identified how to construct a semester course in terms of core material and additional topics for five different types of courses:

1. *Traditional:* emphasizes solution methods
2. *Qualitative:* emphasizes theory of ODEs and properties of solutions
3. *Numerical:* emphasizes algorithms for numerical estimation and control of error
4. *Modeling:* emphasizes applications
5. *General:* a sampling of the first four topics

Other topics can be constructed to suit the instructor's agenda.

Chapter	Core Material	Additional Topics				
		Traditional	Qualitative	Numerical	Modeling	General
1	1.1–1.5				1.6	
2	2.1, 2.2	2.3–2.5				
3			3.1, 3.2	3.1, 3.3	3.1, 3.2	3.1, 3.2
4	4.1–4.5	4.7	4.6		4.6, 4.7	4.6, 4.7
5	5.1		5.4	5.2, 5.3		5.2, 5.4
6		6.1–6.4			6.1–6.6	6.1–6.4
7		7.1–7.3				7.1, 7.2
8	8.1–8.3				8.5	
9			9.2, 9.3	9.1	9.1	9.1, 9.2

Note: Section 1.6 is to be skimmed first and later read as needed in subsequent sections.



Section Dependencies

Technology Aids

Desktop PCs and workstations allow us to do what was unimaginable barely a dozen years ago. Software such as *Macsyma, Maple,* and *Mathematica* is capable of performing most of the symbolic, graphical, and numerical operations described in this book. These programs, known as computer algebra systems (CAS), treat ODEs symbolically. Software such as *Matlab* is primarily used for numerical and graphical operations. (With the exception of some graphs in Section 5.1 that were made with Lascaux Graphics *Fields & Operators,* every graph in this book was produced using either *Maple* or *Matlab*.) These programs are available on MS Windows, Macintosh, and X-Windows platforms. Additionally, *DIFF-E-Q,* the author's MS DOS program for direction fields, phase portraits, and numerical approximations, is available for downloading via ftp from either math.gmu.edu /pub/saperstone or ftp.gmu.edu /math/ssaperst.

Acknowledgments

I cannot heap enough praise on my copyeditor, Susan Gerstein. She read each line of the manuscript not only for grammatical correctness, but for content as well. Because she is trained in mathematics, Susan was able to read the material from a student's point of view and anticipate potential pitfalls. Notation, phrasing, exposition, organization, and formatting are improved as a result of her input. In addition to adding clarity to confusing parts of the manuscript, she pointed out numerous mathematical errors. I want to acknowledge two others: my production editor Kirk Bomont at Brooks/Cole for keeping track of all my alterations, and Ed Rose of Visual Graphic Systems for his faithful rendering of my graphs.

The following reviewers read portions of the manuscript and provided useful feedback during the development of the book: Ed Adams, Adams State College; Linda Allen, Texas Tech University; David C. Buchthal, University of Akron; Frederick Carter, St. Mary's University; M. Hilary Davies, University of Alaska; Michael Ecker, Pennsylvania State University–Wilkes-Barre; Sherif El-Helaly, The Catholic University of America; Newman Fisher, San Francisco State University; Jim Fryxell, College of Lake County; Ronald Guenther, Oregon State University; Donna K. Hafner, Mesa State College; Willy Hereman, Colorado School of Mines; Palle Jorgensen, University of Iowa; Gerald Junevicus, Eckard College; C. J. Knickerbocker, St. Lawrence University; Thomas Kudzma, University of Massachusetts–Lowell; Melvin D. Lax, California State University–Long Beach; Jose L. Menaldi, Wayne State University; Stephen Merrill, Marquette University; Jack Narayan, SUNY–Oswego; Francis J. Narcowich, Texas A&M University; William Radulovich, Florida CC–Kent Campus; Richard Rockwell, Pacific Union College; David Rollins, University of Central Florida; Michel Smith, Auburn University.

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My family has stood by me throughout the years as the book inched its way to print. My wife, Barbara, endured the many hours I spent in the solitude of my manuscript. My children, Amy, Max, and Jenny, helped with some of the graphics and checked manuscript pages for errors. This book would not be possible without their support.

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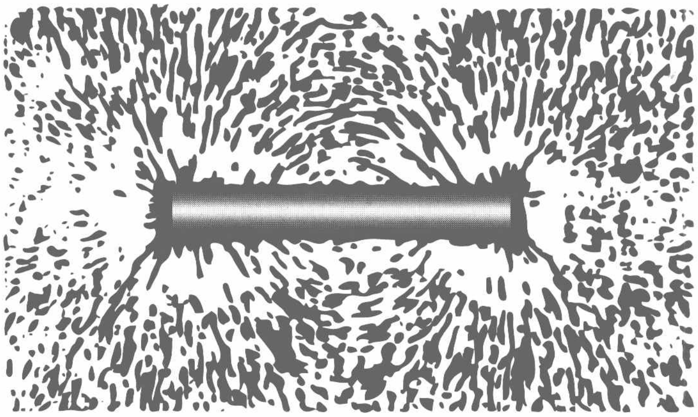
CHAPTER

1 INTRODUCTION



1.1 EXAMPLES OF ODEs

Many systems that undergo changes of state may be described mathematically by an *ordinary differential equation.* In our desire to understand and control our world, we have learned how to use ordinary differential equations to model such diverse phenomena as atmospheric turbulence and epidemiology.

> **DEFINITION** ODE
>
> An **ordinary differential equation (ODE)** is an equation that relates an independent variable, an unknown function of the independent variable, and one or more derivatives of the function.

Here are some typical ODEs:

a. $\dfrac{dx}{dt} = -4x$

b. $\dfrac{dx}{dt} = 0.6x - 0.04x^2$

c. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 6x = e^{-0.2t}\cos t$

d. $t\dfrac{d^2x}{dt^2} = \left(\dfrac{dx}{dt}\right)^2$

e. $\begin{cases} \dfrac{d^2x}{dt^2} = 4y^2 - t^2 + 2 \\ \dfrac{dy}{dt} = -2x + y + e^{-t} \end{cases}$

In these equations, we have chosen to represent the unknown functions by $x(t)$ and $y(t)$. There is nothing special about the choice of the symbol $x(t)$; we could have used and will use the symbols $y(x)$, $u(v)$, etc., for the unknown functions. However, we must be aware, say in the case of $y(x)$, that y is the dependent variable and x is the independent variable. So, for instance, equation (d) may also be expressed

$$x\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

With an occasional exception (as just demonstrated), we adopt the convention of letting t be the independent variable and x, y the dependent variables.

What do we do with an ODE now that we have one? We try to solve it by finding the unknown function $x = x(t)$. A substantial portion of this book is devoted to techniques for solving ODEs. Though most ODEs have solutions, we frequently find that a solution cannot be written down explicitly. However, it's possible to obtain a lot of information about a solution of an ODE without finding that solution in the first place. For example, we will see how to make a sketch of a solution of an ODE, how to determine when a solution has a limiting value as $t \to \infty$, and learn how to calculate such limiting values, *ALL WITHOUT EVER SOLVING THE ODE!*

The solution of an ODE determines a value (or a vector) for the state of a system at each point in time. The great advantage of an ODE formulation of a system is that the future evolution of the system depends only upon the value(s) of the current state and not upon its history. That is, the future states are completely determined by the present state.

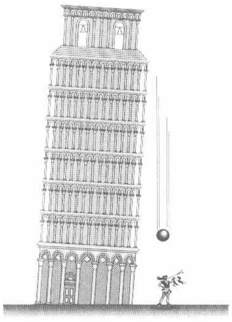
Ordinary differential equations have been around since the development of calculus (by Isaac Newton, 1642–1727, and Gottfried Leibniz, 1646–1716). It should not be surprising that many problems in physics, engineering, and, of late, biology, economics, and even sports can be formulated as ODEs. The examples that follow provide a sampling of the variety of ODEs that occur in practice. Do not get bogged down in trying to understand fully how these equations were derived or how they are solved. Rather, just browse through them and whet your appetite. Each is discussed in greater detail at some later point in the book.

Some Ordinary Differential Equations from the Real World

It is important to note that the following ODEs may appear to be simplistic models of the phenomena they purport to describe. Yet in all instances they convey the essential aspects of the behavior of the examples.

EXAMPLE 1.1 **Falling body (separable equation: Section 1.3)**

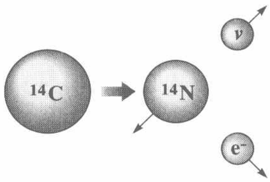
$$\frac{d^2y}{dt^2} = g$$



The symbol y denotes the distance an object has fallen during t units of time. The constant g is the acceleration due to gravity. ■

EXAMPLE 1.2 **Radioactive decay (separable equation: Sections 1.3, 2.6)**

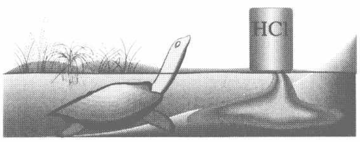
$$\frac{dN}{dt} = -\lambda N$$



The symbol N denotes the number of carbon 14 atoms in a sample of material at time t. The parameter $\lambda > 0$ is a measure of how fast the atoms decay. ■

EXAMPLE 1.3 **Pollution (linear equation: Section 2.1)**

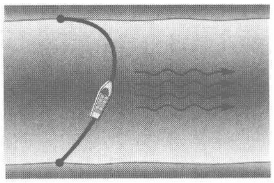
$$\frac{dx}{dt} = r_{IN} - \left(\frac{r_{OUT}}{V_0 + (r_{IN} - r_{OUT})t}\right)x$$



Hydrochloric acid (HCl) accidentally leaks from a storage tank into a spring-fed lake that in turn feeds a single stream. By the time the acid reaches the stream, the acid is uniformly mixed with the lake water. The variable x represents the amount of HCl in the lake at t units of time after the spill. The parameters are: r_{IN}, the rate at which the acid flows into the lake; r_{OUT}, the rate at which the (contaminated) water leaves the lake (by flowing into the stream or by evaporation); and V_0, the initial volume of water in the lake. ■

EXAMPLE 1.4 **Pursuit (homogeneous equation: Section 2.2)**

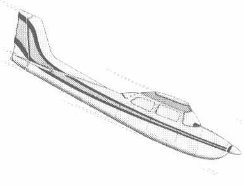
$$\frac{dy}{dx} = \frac{Vy - W\sqrt{x^2 + y^2}}{Vx}$$



The variables x and y refer to the coordinates of a ferryboat that is crossing a river. The parameter V is the boat's speed in still water and the parameter W is the speed of the river current. ■

EXAMPLE 1.5 **Aircraft pull-up from a dive (exact equation: Section 2.3)**

$$\frac{dv}{d\theta} = -\frac{gv \sin\theta}{kv^2 - g\cos\theta}$$

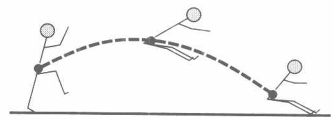


The variable v denotes the aircraft's airspeed and θ denotes the flight-path angle. The constant g is the acceleration due to gravity, and k is a combination of a number of aircraft structure constants. ■

EXAMPLE 1.6

Long jump (reduction-of-order: Section 2.5)

$$m\frac{d^2x}{dt^2} = -c_D A\rho\left(\frac{dx}{dt}\right)^2$$

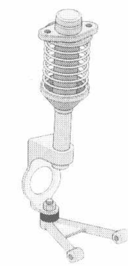


The variable x is the distance traveled by the jumper t units of time after becoming airborne. There are a number of parameters: m is the jumper's mass, A is the area of the vertical cross section the jumper's body presents to the air, ρ is the density of the air, and c_D is the drag coefficient.[1] ■

EXAMPLE 1.7

MacPherson strut (linear second-order equation with constant coefficients: Sections 1.2, 1.6, 4.1, 4.3, and 4.7)

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = f(t)$$

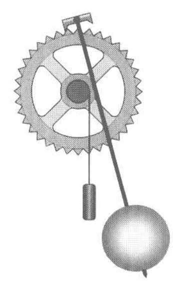


The variable y denotes the displacement at time t of an automobile frame from its rest position. The parameter m is the mass supported by the strut; c is a measure of the damping effect of the shock absorber; and k is the spring constant. The function $f(t)$ represents an external force acting on the strut (such as a road bump). ■

EXAMPLE 1.8

Pendulum clock (Laplace transforms: Section 6.6)

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + \frac{g}{L}\theta = A\delta(\theta)$$



A bob of mass m is attached to a weightless rigid rod of length L. The other end of the rod pivots about a fixed support. The bob is constrained to swing in a vertical plane. The variable θ measures the angle the rod makes with the vertical. An escape wheel drives the hands of the clock through

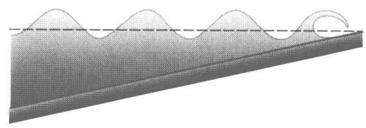
[1] Though the current world record for the long jump was set in 1991 by Mike Powell (29 ft $4\frac{1}{2}$ in.), considerable interest was focused for years on Bob Beamon's world-record jump in the 1968 Olympics in Mexico City. Beamon's jump exceeded the previous world record by over 1 ft 9 in. The mile-high altitude was thought to be a factor in Beamon's feat. Thus the parameter ρ is singled out in order to study the effect of altitude (and body posture) on long-jump performance. The air density (and hence the atmospheric pressure) in Mexico City is 80% of that at sea level. Normally ρ is absorbed by the term c_D.

a sequence of gears. The wheel is mounted on a spindle that rotates as a result of a torque created by a hanging weight. The motion of the escape wheel is stopped by a toothed anchor that rocks back and forth with the pendulum rod. The teeth are designed so that each time the rod swings through the vertical, the escape wheel exerts a small impulse on the anchor, thereby giving the bob an extra push to overcome the friction in the system. The term $A\delta(\theta)$ represents this impulse. The parameter b represents the frictional force and g is a constant, the acceleration due to gravity. ■

EXAMPLE 1.9

Ocean waves (linear second-order equation with nonconstant coefficients: Section 7.5)

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{\omega^2}{\alpha g}y=0$$

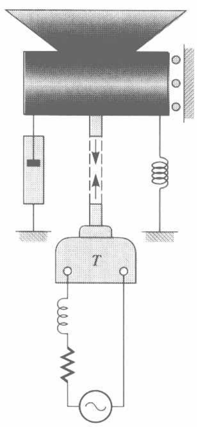


The variable y denotes the height of a wave at the beach or in a wave pool and x its distance from the shoreline or edge. The parameter ω is the frequency of the incoming wave; α is the slope of the ocean or pool floor; and g is the acceleration due to gravity. ■

EXAMPLE 1.10

Loudspeaker (system of linear equations: Section 8.5)

$$\begin{cases}\dfrac{dx}{dt}=y\\[2ex] \dfrac{dy}{dt}=-\dfrac{k}{m}x-\dfrac{c}{m}y+\dfrac{T}{m}i\\[2ex] \dfrac{di}{dt}=-\dfrac{T}{L}y-\dfrac{R}{L}i+E(t)\end{cases}$$

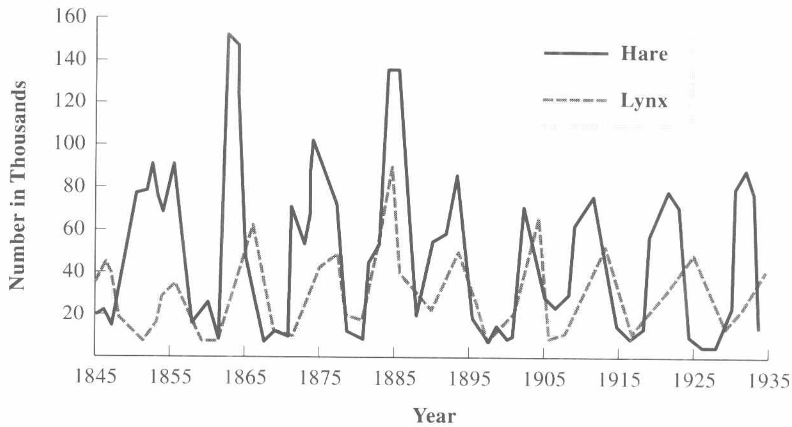


A time-varying voltage source $E(t)$ (typically an audio amplifier) drives a moving-coil transducer T, which in turn causes the speaker diaphragm to vibrate. (The transducer converts electrical energy to mechanical energy.) The variable x denotes the displacement of the speaker diaphragm from equilibrium; y denotes the velocity of the diaphragm. Flexible lead-in wires from the voltage source carry a time-varying current i to the transducer. Internal electrical resistance and self-inductance of the transducer are denoted by R and L, respectively. The motion of the speaker of mass m is modeled as a damped mass–spring system with damping coefficient c and spring constant k. ■

EXAMPLE 1.11

Predator–prey (system of nonlinear equations: Sections 8.1, 9.3)

$$\begin{cases}\dfrac{dx}{dt}=\alpha x-\beta xy\\[2ex] \dfrac{dy}{dt}=-\gamma y+\delta xy\end{cases}$$



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The variables x and y denote the populations of hare and lynx, respectively, in Canada at time t. The parameters are α, the growth rate of the hare in the absence of any lynx, and γ, the death rate of the lynx in the absence of any hare. The quantities β and δ are measures of hare–lynx interactions. ■

Some ODE Terminology

The **order** of an ODE is the order of the highest derivative that appears in the equation. For instance, the order of equation (a) at the opening of this section is 1, the order of equation (c) is 2, and the order of equation (d) is also 2. The order of a *system* of ODEs is the order of the highest-order derivative that appears in any equation of the system. Thus the order of the system (e) is 2. The definition of an ODE given at the start of the section may be expressed symbolically as follows:

DEFINITION General Form of an nth-Order ODE

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}\right) = 0$$

Systems of ODEs, such as equation (e) on p. 1 or Examples 1.10 and 1.11, are comprised of two or more equations, though not all necessarily of the same order. The treatment of systems of ODEs is postponed to Chapters 8 and 9. Until then we consider only single ODEs.

To illustrate the meaning of the general form, we express some of the equations we have seen in terms of a function F.

$$\frac{dx}{dt} = 0.6x - 0.04x^2 \qquad F\left(t, x, \frac{dx}{dt}\right) = \frac{dx}{dt} - 0.6x + 0.04x^2 = 0$$

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad F\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

$$m\frac{d^2x}{dt^2} = -c_D A\rho\left(\frac{dx}{dt}\right)^2 \qquad F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) = m\frac{d^2x}{dt^2} + c_D A\rho\left(\frac{dx}{dt}\right)^2 = 0$$