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Jürgen Jost

Riemannian Geometry and Geometric Analysis

Fourth Edition

黎曼几何和几何分析

第4版

Springer

世界图书出版公司
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With 14 Figures

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图书在版编目（CIP）数据

黎曼几何和几何分析：第4版=Riemannian Gemetry and Geometric Analysis 4th ed.：英文/（德）约斯特（Jost,J.）著.—4版—北京：世界图书出版公司北京公司，2008.3
ISBN 978-7-5062-9192-7

Ⅰ.黎… Ⅱ.约… Ⅲ.黎曼几何—英文 Ⅳ.0186.12

中国版本图书馆CIP数据核字（2008）第022306号

书　　名：Riemannian Geometry and Geometric Analysis 4th ed.
作　　者：Jurgen Jost
中 译 名：黎曼几何和几何分析 第4版
责任编辑：高蓉 刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街137号 100010）
联系电话：010-64015659, 64038348
电子信箱：kjsk@vip.sina.com

开　　本：24开
印　　张：24.5
版　　次：2008年03月第1次印刷
版权登记：图字:01-2008-0985
书　　号：978-7-5062-9192-7 / O・583　　　定　　价：68.00元

**Dedicated to Shing-Tung Yau,
for so many discussions about
mathematics and Chinese culture**

Preface

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts (geodesics, connections, curvature, ...) and objectives, in particular to understand certain classes of (compact) Riemannian manifolds defined by curvature conditions (constant or positive or negative curvature, ...). By way of contrast, geometric analysis is a perhaps somewhat less systematic collection of techniques, for solving extremal problems naturally arising in geometry and for investigating and characterizing their solutions. It turns out that the two fields complement each other very well; geometric analysis offers tools for solving difficult problems in geometry, and Riemannian geometry stimulates progress in geometric analysis by setting ambitious goals.

It is the aim of this book to be a systematic and comprehensive introduction to Riemannian geometry and a representative introduction to the methods of geometric analysis. It attempts a synthesis of geometric and analytic methods in the study of Riemannian manifolds.

The present work is the fourth edition of my textbook on Riemannian geometry and geometric analysis. It has developed on the basis of several graduate courses I taught at the Ruhr-University Bochum and the University of Leipzig. Besides several smaller additions, reorganizations, corrections (I am grateful to J.Weber and P.Hinow for useful comments), and a systematic bibliography, the main new features of the present edition are a systematic introduction to Kähler geometry and the presentation of additional techniques from geometric analysis.

Let me now briefly describe the contents:

In the first chapter, we introduce the basic geometric concepts, like differentiable manifolds, tangent spaces, vector bundles, vector fields and one-parameter groups of diffeomorphisms, Lie algebras and groups and in particular Riemannian metrics. We also derive some elementary results about geodesics.

The second chapter introduces de Rham cohomology groups and the essential tools from elliptic PDE for treating these groups. In later chapters, we shall encounter nonlinear versions of the methods presented here.

The third chapter treats the general theory of connections and curvature.

In the fourth chapter, we introduce Jacobi fields, prove the Rauch comparison theorems for Jacobi fields and apply these results to geodesics.

These first four chapters treat the more elementary and basic aspects of the subject. Their results will be used in the remaining, more advanced chapters that are essentially independent of each other.

The fifth chapter treats symmetric spaces as important examples of Riemannian manifolds in detail.

The sixth chapter is devoted to Morse theory and Floer homology.

The seventh chapter treats variational problems from quantum field theory, in particular the Ginzburg-Landau and Seiberg-Witten equations. The background material on spin geometry and Dirac operators is already developed in earlier chapters.

In the eighth chapter, we treat harmonic maps between Riemannian manifolds. We prove several existence theorems and apply them to Riemannian geometry. The treatment uses an abstract approach based on convexity that should bring out the fundamental structures. We also display a representative sample of techniques from geometric analysis.

A guiding principle for this textbook was that the material in the main body should be self contained. The essential exception is that we use material about Sobolev spaces and linear elliptic PDEs without giving proofs. This material is collected in Appendix A. Appendix B collects some elementary topological results about fundamental groups and covering spaces.

Also, in certain places in Chapter 6, we do not present all technical details, but rather explain some points in a more informal manner, in order to keep the size of that chapter within reasonable limits and not to loose the patience of the readers.

We employ both coordinate-free intrinsic notations and tensor notations depending on local coordinates. We usually develop a concept in both notations while we sometimes alternate in the proofs. Besides my not being a methodological purist, the reasons for often prefering the tensor calculus to the more elegant and concise intrinsic one are the following. For the analytic aspects, one often has to employ results about (elliptic) partial differential equations (PDEs), and in order to check that the relevant assumptions like ellipticity hold and in order to make contact with the notations usually employed in PDE theory, one has to write down the differential equation in local coordinates. Also, recently, manifold and important connections have been established between theoretical physics and our subject. In the physical literature, tensor notation is usually employed, and therefore familiarity with that notation is necessary to explore those connections that have been found to be stimulating for the development of mathematics, or promise to be so in the future.

As appendices to most of the paragraphs, we have written sections with the title "Perspectives". The aim of those sections is to place the material in a broader context and explain further results and directions without detailed proofs. The material of these Perspectives will not be used in the main body of the text. At the end of each chapter, some exercises for the reader are given.

We assume of the reader sufficient perspicacity to understand our system of numbering and cross-references without further explanation.

The development of the mathematical subject of Geometric Analysis, namely the investigation of analytical questions arising from a geometric context and in turn the application of analytical techniques to geometric problems, is to a large extent due to the work and the influence of Shing-Tung Yau. This book, like its previous editions, is dedicated to him.

Jürgen Jost

Contents

1. Foundational Material

1.1 Manifolds and Differentiable Manifolds

A *topological space* is a set M together with a family $\mathcal{O}$ of subsets of M satisfying the following properties:

(i) $\Omega_1, \Omega_2 \in \mathcal{O} \Rightarrow \Omega_1 \cap \Omega_2 \in \mathcal{O}$

(ii) For any index set A:

$$(\Omega_\alpha)_{\alpha \in A} \subset \mathcal{O} \Rightarrow \bigcup_{\alpha \in A} \Omega_\alpha \in \mathcal{O}$$

(iii) $\emptyset, M \in \mathcal{O}$

The sets from $\mathcal{O}$ are called *open*. A topological space is called *Hausdorff* if for any two distinct points $p_1, p_2 \in M$ there exists open sets $\Omega_1, \Omega_2 \in \mathcal{O}$ with $p_1 \in \Omega_1, p_2 \in \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset$. A covering $(\Omega_\alpha)_{\alpha \in A}$ (A an arbitrary index set) is called *locally finite* if each $p \in M$ has a neighborhood that intersects only finitely many Ω_α. M is called *paracompact* if any open covering possesses a locally finite refinement. This means that for any open covering $(\Omega_\alpha)_{\alpha \in A}$ there exists a locally finite open covering $(\Omega'_\beta)_{\beta \in B}$ with

$$\forall \beta \in B \; \exists \alpha \in A : \Omega'_\beta \subset \Omega_\alpha.$$

A map between topological spaces is called *continuous* if the preimage of any open set is again open. A bijective map which is continuous in both directions is called a *homeomorphism*.

Definition 1.1.1 A *manifold* M of *dimension* d is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomorphic to an open subset Ω of $\mathbb{R}^d$. Such a homeomorphism

$$x : U \to \Omega$$

is called a *(coordinate) chart*.

An *atlas* is a family $\{U_\alpha, x_\alpha\}$ of charts for which the U_α constitute an open covering of M.

Remarks.

1) A point $p \in U_\alpha$ is determined by $x_\alpha(p)$; hence it is often identified with $x_\alpha(p)$. Often, also the index α is omitted, and the components of $x(p) \in \mathbb{R}^d$ are called *local coordinates* of p.

2) Any atlas is contained in a maximal one, namely the one consisting of all charts compatible with the original one.

Definition 1.1.2 An atlas $\{U_\alpha, x_\alpha\}$ on a manifold is called *differentiable* if all chart transitions

$$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ (in case $U_\alpha \cap U_\beta \neq \emptyset$). A maximal differentiable atlas is called a differentiable structure, and a *differentiable manifold* of dimension d is a manifold of dimension d with a differentiable structure. From now on, all atlases are supposed to be differentiable. Two atlases are called compatible if their union is again an atlas. In general, a chart is called compatible with an atlas if adding the chart to the atlas yields again an atlas. An atlas is called maximal if any chart compatible with it is already contained in it.

Remarks.

1) Since the inverse of $x_\beta \circ x_\alpha^{-1}$ is $x_\alpha \circ x_\beta^{-1}$, chart transitions are differentiable in both directions, i.e. diffeomorphisms.

2) One could also require a weaker differentiability property than C^∞.

3) It is easy to show that the dimension of a differentiable manifold is uniquely determined. For a general, not differentiable manifold, this is much harder.

4) Since any differentiable atlas is contained in a maximal differentiable one, it suffices to exhibit some differentiable atlas if one wants to construct a differentiable manifold.

Definition 1.1.3 An atlas for a differentiable manifold is called *oriented* if all chart transitions have positive functional determinant. A differentiable manifold is called *orientable* if it possesses an oriented atlas.

It is customary to write the Euclidean coordinates of $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ open, as

$$x = (x^1, \ldots, x^d), \tag{1.1.1}$$

and these then are considered as local coordinates on our manifold M when $x : U \to \Omega$ is a chart.

Example.

1) The *sphere* $S^n := \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1\}$ is a differentiable manifold of dimension n. Charts can be given as follows: On $U_1 := S^n \backslash \{(0, \ldots, 0, 1)\}$ we put

$$f_1(x^1, \ldots, x^{n+1}) := (f_1^1(x^1, \ldots, x^{n+1}), \ldots, f_1^n(x^1, \ldots, x^{n+1})) := \left(\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}}\right)$$

and on $U_2 := S^n \backslash \{(0, \ldots, 0, -1)\}$

$$f_2(x^1, \ldots, x^{n+1}) := (f_2^1(x^1, \ldots, x^{n+1}), \ldots, f_2^n(x^1, \ldots, x^{n+1})) := \left(\frac{x^1}{1 + x^{n+1}}, \ldots, \frac{x^n}{1 + x^{n+1}}\right).$$

2) Let $w_1, w_2, \ldots, w_n \in \mathbb{R}^n$ be linearly independent. We consider $z_1, z_2 \in \mathbb{R}^n$ as equivalent if there are $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ with

$$z_1 - z_2 = \sum_{i=1}^{n} m_i w_i$$

Let π be the projection mapping $z \in \mathbb{R}^n$ to its equivalence class. The *torus* $T^n := \pi(\mathbb{R}^n)$ can then be made a differentiable manifold (of dimension n) as follows: Suppose Δ_α is open and does not contain any pair of equivalent points. We put

$$U_\alpha := \pi(\Delta_\alpha)$$
$$z_\alpha = (\pi_{|\Delta_\alpha})^{-1}.$$

3) The preceding examples are compact. Of course, there exist also non-compact manifolds. The simplest example is $\mathbb{R}^d$. In general, any open subset of a (differentiable) manifold is again a (differentiable) manifold.

4) If M and N are differentiable manifolds, the Cartesian product $M \times N$ also naturally carries the structure of a differentiable manifold. Namely, if $\{U_\alpha, x_\alpha\}_{\alpha \in A}$ and $\{V_\beta, y_\beta\}_{\beta \in B}$ are atlases for M and N, resp., then $\{U_\alpha \times V_\beta, (x_\alpha, y_\beta)\}_{(\alpha,\beta) \in A \times B}$ is an atlas for $M \times N$ with differentiable chart transitions.

Definition 1.1.4 A map $h : M \to M'$ between differentiable manifolds M and M' with charts $\{U_\alpha, x_\alpha\}$ and $\{U'_\alpha, x'_\alpha\}$ is called *differentiable* if all maps $x'_\beta \circ h \circ x_\alpha^{-1}$ are differentiable (of class C^∞, as always) where defined.
Such a map is called a *diffeomorphism* if bijective and differentiable in both directions.

For purposes of differentiation, a differentiable manifold locally has the structure of Euclidean space. Thus, the differentiability of a map can be tested in local coordinates. The diffeomorphism requirement for the chart transitions then guarantees that differentiability defined in this manner is a consistent notion, i.e. independent of the choice of a chart.

Remark. We want to point out that in the context of the preceding definitions, one cannot distinguish between two homeomorphic manifolds nor between two diffeomorphic differentiable manifolds.

When looking at Definitions 1.1.2, 1.1.3, one may see a general pattern emerging. Namely, one can put any type of restriction on the chart transitions, for example, require them to be affine, algebraic, real analytic, conformal, Euclidean volume preserving,..., and thereby define a class of manifolds with that particular structure. Perhaps the most important example is the notion of a complex manifold. We shall need this, however, only at certain places in this book, namely in 5.1, 5.2.

Definition 1.1.5 A *complex manifold* of complex dimension d ($\dim_{\mathbb{C}} M = d$) is a differentiable manifold of (real) dimension $2d$ ($\dim_{\mathbb{R}} M = 2d$) whose charts take values in open subsets of $\mathbb{C}^d$ with *holomorphic* chart transitions.

In the case of a complex manifold, it is customary to write the coordinates of $\mathbb{C}^d$ as

$$z = (z^1, \ldots, z^d), \qquad \text{with} \qquad z^j = x^j + iy^j, \tag{1.1.2}$$

with $i := \sqrt{-1}$, that is, use $(x^1, y^1, \ldots, x^d, y^d)$ as Euclidean coordinates on $\mathbb{R}^{2d}$. We then also put

$$z^{\bar{j}} := x^j - iy^j.$$

The requirement that the chart transitions $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \to z_\beta(U_\alpha \cap U_\beta)$ be holomorphic then is expressed as

$$\frac{\partial z_\beta^j}{\partial z_\alpha^{\bar{k}}} = 0 \tag{1.1.3}$$

for all j, k where

$$\frac{\partial}{\partial z^{\bar{k}}} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right). \tag{1.1.4}$$

We also observe that a complex manifold is always orientable because holomorphic maps always have a positive functional determinant.

We conclude this section with a useful technical result.

Lemma 1.1.1 *Let M be a differentiable manifold, $(U_\alpha)|_{\alpha \in A}$ an open covering. Then there exists a partition of unity, subordinate to (U_α). This means* *that there exists a locally finite refinement* $(V_\beta)_{\beta\in B}$ *of* (U_α) *and* C_0^∞ *(i.e.* C^∞ *functions* φ_β *with* $\{x \in M : \varphi_\beta(x) \neq 0\}$ *having compact closure) functions* $\varphi_\beta : M \to \mathbb{R}$ *with*

(i) $\quad supp\, \varphi_\beta \subset V_\beta$ *for all* $\beta \in B$.

(ii) $\quad 0 \leq \varphi_\beta(x) \leq 1$ *for all* $x \in M, \beta \in B$.

(iii) $\quad \sum_{\beta\in B} \varphi_\beta(x) = 1$ *for all* $x \in M$.

Note that in (iii), there are only finitely many nonvanishing summands at each point since only finitely many φ_β *are nonzero at any given point because the covering* (V_β) *is locally finite.*

Proof. See any advanced textbook on Analysis, e.g. J.Jost, Postmodern Analysis, 3rd ed., Springer, 2005. □

Perspectives. Like so many things in Riemannian geometry, the concept of a differentiable manifold was in some vague manner implicitly contained in Bernhard Riemann's habilitation address "Über die Hypothesen, welche der Geometrie zugrunde liegen", reprinted in [254]. The first clear formulation of that concept, however, was given by H. Weyl[252].

The only one dimensional manifolds are the real line and the unit circle S^1, the latter being the only compact one. Two dimensional compact manifolds are classified by their genus and orientability character. In three dimensions, there exists a program by Thurston[243, 244] about the possible classification of compact three-dimensional manifolds. References for the geometric approach to this classification will be given in the Survey on Curvature and Topology after Chapter 4 below. – In higher dimensions, the plethora of compact manifolds makes a classification useless and impossible.

In dimension at most three, each manifold carries a unique differentiable structure, and so here the classifications of manifolds and differentiable manifolds coincide. This is not so anymore in higher dimensions. Milnor[180, 181] discovered exotic 7-spheres, i.e. differentiable structures on the manifold S^7 that are not diffeomorphic to the standard differentiable structure exhibited in our example. Exotic spheres likewise exist in higher dimensions. Kervaire[156] found an example of a manifold carrying no differentiable structure at all.

In dimension 4, the understanding of differentiable structures owes important progress to the work of Donaldson. He defined invariants of a differentiable 4-manifold M from the space of selfdual connections on principal bundles over it. These concepts will be discussed in more detail in §3.2.

In particular, there exist exotic structures on $\mathbb{R}^4$. A description can e.g. be found in [79].

1.2 Tangent Spaces

Let $x = (x^1, \dots, x^d)$ be Euclidean coordinates of $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ open, $x_0 \in \Omega$. The tangent space of Ω at the point x_0,

$$T_{x_0}\Omega$$

is the space $\{x_0\} \times E$, where E is the d-dimensional vector space spanned by the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$. Here, $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$ are the partial derivatives at the point x_0. If $\Omega \subset \mathbb{R}^d, \Omega' \subset \mathbb{R}^c$ are open, and $f : \Omega \to \Omega'$ is differentiable, we define the *derivative* $df(x_0)$ for $x_0 \in \Omega$ as the induced linear map between the tangent spaces

$$\begin{aligned} df(x_0) : T_{x_0}\Omega &\to T_{f(x_0)}\Omega' \\ v = v^i \frac{\partial}{\partial x^i} &\mapsto v^i \frac{\partial f^j}{\partial x^i}(x_0) \frac{\partial}{\partial f^j} \end{aligned}$$

Here and in the sequel, we use the *Einstein summation convention:* An index occuring twice in a product is to be summed from 1 up to the space dimension. Thus, $v^i \frac{\partial}{\partial x^i}$ is an abbreviation for

$$\sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i},$$

$v^i \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial f^j}$ stands for

$$\sum_{i=1}^{d} \sum_{j=1}^{c} v^i \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial f^j}.$$

In the previous notations, we put

$$T\Omega := \Omega \times E \cong \Omega \times \mathbb{R}^d.$$

Thus, $T\Omega$ is an open subset of $\mathbb{R}^d \times \mathbb{R}^d$, hence in particular a differentiable manifold.

$$\begin{aligned} \pi : T\Omega &\to \Omega \quad \text{(projection onto the first factor)} \\ (x, v) &\mapsto x \end{aligned}$$

is called a tangent bundle of Ω. $T\Omega$ is called the total space of the tangent bundle.

Likewise, we define

$$\begin{aligned} df :& T\Omega \to T\Omega' \\ &(x, v^i \frac{\partial}{\partial x^i}) \mapsto (f(x), v^i \frac{\partial f^j}{\partial x^i}(x) \frac{\partial}{\partial f^j}) \end{aligned}$$

Instead of

$$df(x, v)$$

we write

$$df(x)(v).$$

If in particular, $f : \Omega \to \mathbb{R}$ is a differentiable function, we have for $v = v^i \frac{\partial}{\partial x^i}$

$$df(x)(v) = v^i \frac{\partial f}{\partial x^i}(x) \in T_{f(x)}\mathbb{R} \cong \mathbb{R}.$$

In this case, we often write $v(f)(x)$ in place of $df(x)(v)$ when we want to express that the tangent vector v operates by differentiation on the function f.

Let now M be a differentiable manifold of dimension d, and $p \in M$. We want to define the tangent space of M at the point p. Let $x : U \to \mathbb{R}^d$ be a chart with $p \in U$, U open in M. We say that the tangent space T_pM is represented in the chart x by $T_{x(p)}x(U)$. Let $x' : U' \to \mathbb{R}^d$ be another chart with $p \in U'$, U' open in M. $\Omega := x(U)$, $\Omega' := x'(U')$. The transition map

$$x' \circ x^{-1} : x(U \cap U') \to x'(U \cap U')$$

induces a vector space isomorphism

$$L := d(x' \circ x^{-1})(x(p)) : T_{x(p)}\Omega \to T_{x'(p)}\Omega'.$$

We say that $v \in T_{x(p)}\Omega$ and $L(v) \in T_{x'(p)}\Omega'$ represent the same tangent vector in T_pM. Thus, a tangent vector in T_pM is given by the family of its coordinate representations. This is motivated as follows: Let $f : M \to \mathbb{R}$ be a differentiable function. Assume that the tangent vector $w \in T_pM$ is represented by $v \in T_{x(p)}x(U)$. We then want to define $df(p)$ as a linear map from T_pM to $\mathbb{R}$. In the chart x, let $w \in T_pM$ be represented by $v = v^i \frac{\partial}{\partial x^i} \in T_{x(p)}x(U)$. We then say that

$$df(p)(w)$$

in this chart is represented by

$$d(f \circ x^{-1})(x(p))(v).$$

Now

$$\begin{aligned} d(f \circ x^{-1})(x(p))(v) &= d(f \circ x'^{-1} \circ x' \circ x^{-1})(x(p))(v) \\ &= d(f \circ x'^{-1})(x'(p))(L(v)) \\ &\quad \text{by the chain rule} \\ &= d(f \circ x'^{-1})(x'(p)) \circ d(x' \circ x^{-1})(x(p))(v) \end{aligned}$$