Recursive Methods in Economic Dynamics

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with Edward C. Prescott

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$x_i \to x$	converges
$x_i \uparrow x$	converges from below
$x_i \downarrow x$	converges from above
$(X, \mathscr{X})$	measurable space
$M(X, \mathscr{X})$	space of measurable real-valued functions on $(X, \mathscr{X})$
$M^+(X, \mathscr{X})$	subset of $M(X, \mathscr{X})$ containing nonnegative functions
$B(X, \mathscr{X})$	space of bounded measurable real-valued functions on $(X, \mathscr{X})$
$(X, \mathscr{X}, \mu)$	measure space
$L(X, \mathscr{X}, \mu)$	space of μ-integrable functions on $(X, \mathscr{X})$
$\Lambda(X, \mathscr{X})$	space of probability measures on $(X, \mathscr{X})$
μ-a.e.	except on a set A with $\mu(A) = 0$
$\mu \perp \lambda$	mutually singular
$\mu \ll \lambda$	absolutely continuous with respect to
$\lambda_n \to \lambda$	converges in the total variation norm
$\lambda_n \Rightarrow \lambda$	converges weakly
$\mathscr{A} \times \mathscr{B}$	product σ-algebra

Preface

This book was motivated by our conviction that recursive methods should be part of every economist's set of analytical tools. Applications of these methods appear in almost every substantive area of economics—the theory of investment, the theory of the consumer, search theory, public finance, growth theory, and so on—but neither the methods nor the applications have ever been drawn together and presented in a systematic way. Our goal has been to do precisely this. We have attempted to develop the basic tools of recursive analysis in a systematic, rigorous way, while at the same time stressing the wide applicability of recursive methods and suggesting new areas where they might usefully be exploited.

Our first outlines for the book included a few chapters devoted to mathematical preliminaries, followed by numerous chapters treating the various substantive areas of economics to which recursive methods have been applied. We hoped to keep the technical material to a minimum, by simply citing the existing literature for most of the required mathematical results, and to focus on substantive issues. This plan failed rather quickly, as it soon became apparent that the reader would be required either to take most of the important results on faith or else to keep a dozen mathematics books close at hand and refer to them constantly. Neither approach seemed reasonable, and we were led to make major alterations in the overall structure of the book.

The methods became the organizing principle, and we began to focus on providing a fairly comprehensive, rigorous, and self-contained treatment of the tools and techniques used in recursive analysis. We then found it natural to group applications by the nature of the technical tools involved rather than by their economic substance. Thus Parts II–IV of the book deal with deterministic models, stochastic models, and equilibrium theory, respectively, with substantive applications appearing in all three places. Indeed, many of the applications appear more than once, with different aspects of the same problem treated as the appropriate tools are developed.

Once we had decided to write a book focused on analytical tools rather than on economic substance, the choice of technical level became more important than ever. We wanted the book to be rigorous enough to be useful to researchers and at the same time to be accessible to as wide an audience as possible. In pursuing these twin goals we have aimed for a rigorous and fairly general treatment of the analytical tools, but one that requires relatively little by way of mathematical background. The reader should have had a course in advanced calculus or real analysis and should be comfortable with delta-epsilon arguments. A little background in probability theory is also useful, although not at all essential. The other mathematical topics that arise—and there are a wide variety—are treated in a largely self-contained way.

The most difficult decision we faced was choosing the appropriate level at which to treat probability theory. Our first inclination was to restrict attention to discrete probabilities and continuous densities, but in the end we found that this approach caused more trouble than it saved. We were pleased to find that a relatively small investment in measure theory produced enormous returns. We provide a modest number of definitions and basic results from the abstract theory of measure and integration in Chapter 7, and then draw on them repeatedly in our treatment of stochastic models. The reader will find that this investment yields returns elsewhere as well: measure theory is rapidly becoming the standard language of the economics of uncertainty.

The term *recursive methods* is broad enough to include a variety of interesting topics that might have been included in the book but are not. There is a large literature on linear-quadratic formulations of dynamic problems that, except for examples discussed briefly in Chapters 4 and 9, we ignore. There is also a growing body of expertise on methods for the numerical solution of recursive models that we have not attempted to incorporate in this volume. Although a wide variety of dynamic games can be analyzed by recursive methods, our examples of equilibrium are almost exclusively competitive. We have included a large collection of applications, but we certainly have not exhausted the many applied literatures where recursive methods are being used. Yet these omissions are not, we feel, cause for apology. The book is long enough as it is, and we will certainly not be disappointed if one of the functions it serves is to stimulate the reader to a more serious exploration of some of these closely related areas.

We have tried to write this book in a way that will make it useful for several different types of readers. Those who are familiar with dynamic economic models and have specific questions in mind are invited simply to consult the table of contents and proceed to the particular topics that interest them. We have tried to make chapters and sections sufficiently self-contained so that the book can be used in this way. Primarily, however, the book is directed at the reader with little or no background in dynamic models. The manuscript has, at a variety of stages, been used for graduate-level courses at Chicago, Minnesota, Northwestern, and elsewhere, and we have been gratified with the response from students. The book is about the right length and level for a year-long course for second-year students but can easily be adapted for shorter courses as well. After the introductory material in Chapters 1 and 2, it is probably advisable to cover Chapter 3 in detail, skim Section 4.1, cover Section 4.2 in detail, and then choose a few applications from Chapter 5. For a one-quarter course, there are then several possibilities. One could skip to Chapters 15 and 16, and if time permits, go on to 17 and 18. Alternatively, with measure theory as a prerequisite, one could proceed to Section 8.1, then to Section 9.2, and then to applications from Chapter 10. Covering the required measure theory, Sections 7.0–7.5, takes about three weeks and could be done in a one-semester course.

A consequence of our decision to make the book technically self-contained is that completing it involved a much higher ratio of exposition to new results than any of us had anticipated. Ed Prescott found he did not wish to spend so much of his time away from the research frontier, and so proposed the reduced level of involvement reflected in the phrase "with the collaboration of." However, there is no part of the book that has not benefited from his ideas and contributions.

We are grateful also to many friends and colleagues for their comments and criticism. In particular we thank Andrew Caplin, V. V. Chari, Lars Hansen, Hugo Hopenhayn, Larry Jones, Lars Ljungquist, Rodolfo Manuelli, Masao Ogaki, José Victor Rios-Rull, and José Scheinkman for fruitful discussions. Arthur Kupferman read large portions of the manuscript at an early stage, and his detailed comments enhanced both the

Symbols Used

$x \in X$	element
$A \subseteq B$, $A \subset B$	subset, strict subset
$A \supseteq B$, $A \supset B$	superset, strict superset
$\cup$, $\cap$	union, intersection
$\emptyset$	empty set
$A \backslash B$	difference, defined only if $A \supseteq B$
A^c	complement
$\mathring{A}$, $\bar{A}$	interior, closure
∂A	boundary
χ_A	indicator function
$X \times Y$	Cartesian product
$\mathbf{R}$, $\bar{\mathbf{R}}$	real numbers, extended real numbers
$\mathbf{R}^l$	l-dimensional Euclidean space
$\mathbf{R}^l_+$, $\mathbf{R}^l_{++}$	subspace of $\mathbf{R}^l$ containing nonnegative vectors, strictly positive vectors
(a, b), $[a, b]$	open interval, closed interval
$(a, b]$, $[a, b)$	half-open intervals
$\mathscr{B}$, $\mathscr{B}^l$	Borel subsets of $\mathbf{R}$, of $\mathbf{R}^l$
$\mathscr{B}_X$	Borel subsets of X, defined for $X \in \mathscr{B}^l$
$\rho(x, y)$	distance
$\|x\|$	norm
$C(X)$	space of bounded continuous functions on X
f^+, f^-	positive and negative parts of the function f
$\{x_i\}_{i=1}^n$	finite sequence
$\{x_i\}_{i=1}^\infty$	infinite sequence

To our parents

Contents

PART I

The Recursive Approach