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EDITION

THE FINITE ELEMENT METHOD ITS BASIS & FUNDAMENTALS

有限元方法：基础理论 第7版

O.C. Zienkiewicz, R.L. Taylor and J.Z. Zhu

Elsevier (Singapore) Pte Ltd.

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The Finite Element Method: Its Basis and Fundamentals

Seventh Edition

O.C. Zienkiewicz, CBE, FRS

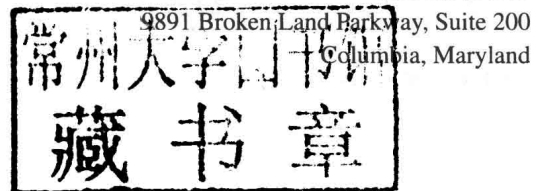
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Butterworth-Heinemann is an imprint of Elsevier



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ISBN:978-1-85617-633-0

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Authorized English language reprint edition published by the
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3 Killiney Road

#08 - 01 Winsland House I

Singapore 239519

Tel: (65) 6349 - 0200

Fax: (65) 6733 - 1817

First Published 2015

2015 年初版

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The Finite Element Method: Its Basis and Fundamentals

Professor O.C. Zienkiewicz, CBE, FRS, FREng died on 2 January 2009. Prior to his death he was Professor Emeritus at the Civil and Computational Engineering Centre, University of Wales, Swansea and previously Director of the Institute for Numerical Methods in Engineering at the University of Wales, Swansea, UK. He also held the UNESCO Chair of Numerical Methods in Engineering at the Technical University of Catalunya, Barcelona, Spain. He was the head of the Civil Engineering Department at the University of Wales, Swansea between 1961 and 1989. During this period he established that department as one of the primary centers of finite element research. In 1968 he became the Founder Editor of the *International Journal for Numerical Methods in Engineering* which still remains today the major journal in this field. The recipient of 27 honorary degrees and many medals, Professor Zienkiewicz was a member of five academies—an honor he received for his many contributions to the fundamental developments of the finite element method. In 1978, he became a Fellow of the Royal Society and the Royal Academy of Engineering. This was followed by his election as a foreign member to the U.S. Academy of Engineering (1981), the Polish Academy of Science (1985), the Chinese Academy of Sciences (1998), and the National Academy of Science, Italy (Accademia dei Lincei) (1999). He published the first edition of this book in 1967 and it remained the only book on the subject until 1971.

Professor R.L. Taylor has more than 50 years' experience in the modeling and simulation of structures and solid continua including eight years in industry. He is Professor of the Graduate School and the Emeritus T.Y. and Margaret Lin Professor of Engineering at the University of California at Berkeley and Corporate Fellow at Dassault Systèmes SIMULIA in Providence, Rhode Island. In 1991 he was elected to membership in the US National Academy of Engineering in recognition of his educational and research contributions to the field of computational mechanics. He is a Fellow of the US Association of Computational Mechanics—USACM (1996) and a Fellow of the International Association of Computational Mechanics—IACM (1998). He has received numerous awards including the Berkeley Citation, the highest honor awarded by the University of California at Berkeley, the USACM John von Neumann Medal, the IACM Gauss–Newton Congress Medal, and a Dr.-Ingenieur ehrenhalber awarded by the Technical University of Hannover, Germany. He has written several computer programs for finite element analysis of structural and non-structural systems, one of which, *FEAP*, is used worldwide in education and research environments. A personal version, *FEAppv*, available at his UC website, is incorporated into this book.

Dr. J.Z. Zhu has more than 30 years' experience in the development of finite element methods. During the last 20 years he has worked in industry where he has been developing commercial finite element software to solve multi-physics problems. Dr Zhu read for his Bachelor of Science degree at Harbin Engineering University and his Master of Science at Tianjin University, both in China. He was awarded his doctoral degree in 1987 from the University of Wales, Swansea, working under the supervision of Professor Zienkiewicz. Dr Zhu is the author of more than 40 technical papers on finite element methods including several on error estimation and adaptive automatic mesh generation. These have resulted in his being named in 2000 as one of the most highly cited researchers for engineering in the world and in 2001 as one of the top 20 most highly cited researchers for engineering in the United Kingdom.

*This book is dedicated to the memory of
Olgierd C. (Olek) Zienkiewicz:
Pioneer, mentor, and close friend.*

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