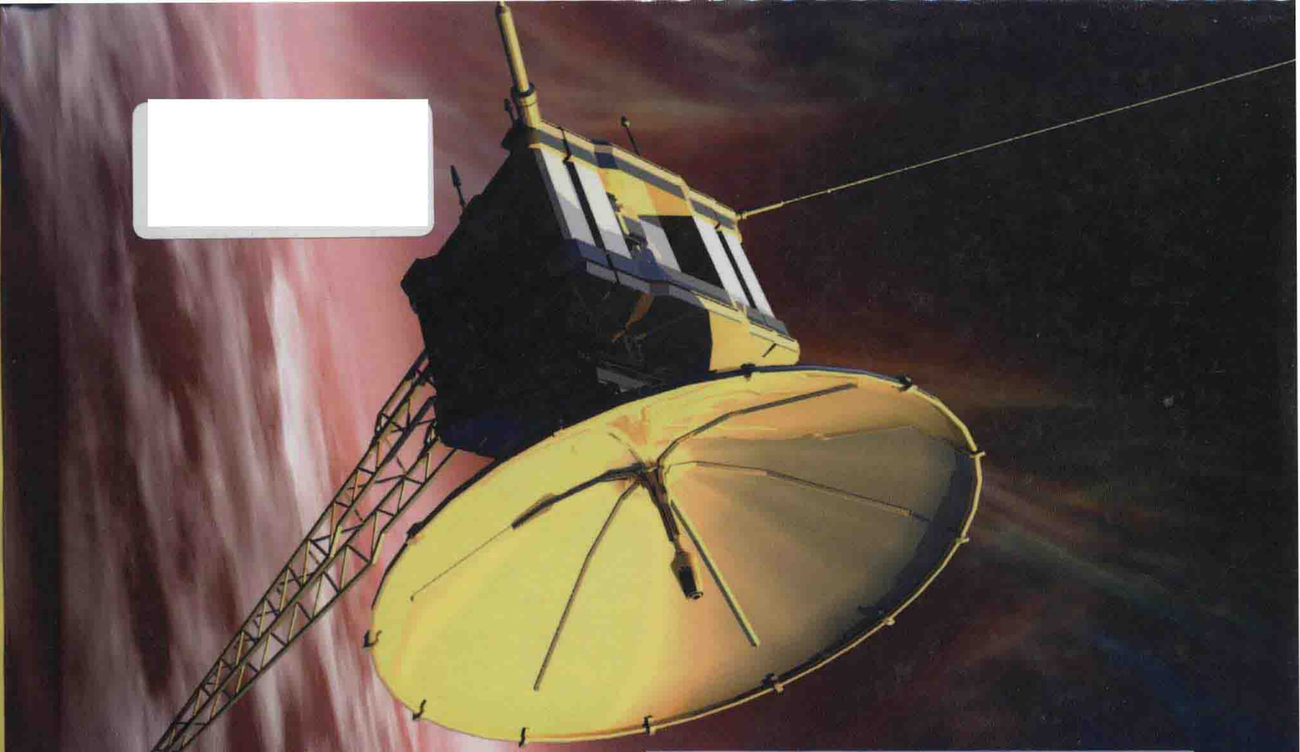

ARUN K. BANERJEE

FLEXIBLE MULTIBODY DYNAMICS

Efficient Formulations and Applications

WILEY

FLEXIBLE MULTIBODY DYNAMICS

EFFICIENT FORMULATIONS AND APPLICATIONS

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This book is dedicated to:

Alpona, my wife who suffered over thirty-five years as I worked at my job and on weekends and nights for publications, and urged me to write this book after I retired, but did not live to see it published;

and

Professor Thomas Kane, my mentor, from whose magnificent books I learned to *do* dynamics, and who started me on the path to published research that this book embodies;

and

Our daughter, Onureena, and son, Abheek, who have been my blessings;

and

Lockheed Martin Space Systems Company and its managers, particularly Dr. Ron Dotson, who gave me a free hand to work on the algorithms reported here, which led to a flexible multibody dynamics code developed with major help from my colleague, Mark Lemak, and several independent formulations.

Preface

This book is based on my published research on deriving computationally efficient equations of motion of multibody systems with rotating, flexible components. It reflects my work of over thirty-five years done mostly at Lockheed Missiles & Space Company, and also at Martin Marietta and Northrop Corporations. The cover of the book depicts two examples of flexible multibody systems: the Galileo spacecraft, which was sent to Jupiter, with its rotating antenna dish on an inertially-fixed base with a deployed truss, and a helicopter in flight. Other examples of multibody systems, apart from the human body itself, are robotic manipulators, a space shuttle deploying a tethered subsatellite, and a ship reeling out a cable to a vehicle doing sea floor mine searches. Formulation of equations of motion is the first step in their simulation-based design.

In this book, I choose to use Kane's method of deriving equations of motion, for two reasons: efficiency in reducing labor of deriving the equations, and simplicity of the final equations due to a choice of variables that the method allows. However, the contribution of the book goes beyond a direct formulation of Kane's equations to more computationally efficient algorithms like block-diagonal and order-n formulations. Another major contribution of this book is in compensating for errors of premature linearization, inherent with the use of vibration modes in large overall motion problems, by using geometric stiffness due to inertia loads.

A highlight of this book is the application of the theory to complex problems. In Chapter 1, I explain Kane's method, first with a simple example and then by applying it to a realistic problem of the dynamics of a three-axis controlled spacecraft with fuel slosh. Presented separately are Kane's method of direct linearization of equation of motion and a method of a posteriori compensation for premature linearization by adding geometric stiffness due to inertia loads; in the Appendix, a guideline for choosing variables that simplify equations of motion is provided. In Chapter 2, Kane's method is used to derive nonlinear dynamical equations for tethered satellite deployment, station-keeping and retrieval, and a problem of impact dynamics of a nose cap during ejection of a parachute for recovery of a booster launching a satellite. The next two chapters cover large overall motion of beams and plates that illustrate the application of Kane's method of direct linearization. Chapter 5 gives a derivation of equations of large overall motion of an arbitrary flexible body, with a method of redeeming prematurely linearized equations by adding motion-induced geometric stiffness. Chapter 6 incorporates the motion-induced geometric stiffness into the dynamics of a system of flexible bodies in large overall motions. Chapter 7 is a review material from structural dynamics, based mainly on the book by Craig, with some additional work on mode selection done at Lockheed. Chapter 8 produces an algorithm for dynamical equations, with block-diagonal mass matrices, used for the Hubble and Next Generation Space telescopes, and to systems with and without structural loops, comparing results with test data for an antenna deployment. Chapter 9 illustrates the power of efficient motion variables in a block-diagonal algorithm, treating multiple loops. Chapter 10 simplifies the block-diagonal formulation to an order-n method for a system of spring-connected rigid rods, to simulate large bending of beams in large overall motion, comparing results with the finite element method; a Fortran code for the formulation is in Appendix B of this book. Chapter 11 uses a variable-n order-n algorithm for deploying a boom from a spacecraft, and a cable from a ship to an underwater vehicle. Chapter 12 covers flexible rocket dynamics.

This book is for readers with backgrounds in rigid body dynamics and structural dynamics. In writing it I was helped by Prof. Paul Mitiguy at Stanford (on efficient variables), Prof. Arun Misra at McGill (on formation flying of tethered satellites), and Dr. John Dickens of Lockheed (on modal truncation vectors and geometric stiffness issues). I thank my Lockheed colleagues: Mark Lemak, who developed a multibody dynamics code from the algorithms given here and produced the results in Chapters 6–9; and David Levinson, whose high praise was a booster for me to write this book. John Dickens provided the structural dynamics codes. Dr. Ron Dotson, a manager at Lockheed, gave me a free hand to develop the algorithms. Dr. Tushar Ghosh of L-3 Communications advised me on current practice, and meticulously edited the book.

Contents

1

Derivation of Equations of Motion

1.1 Available Analytical Methods and the Reason for Choosing Kane's Method

In this book we derive equations of motion for a system of rigid and flexible bodies undergoing large overall motion. Various choices of analytical methods are available for this task, such as Newton-Euler methods, and methods based on D'Alembert's principle together with the principle of virtual work, Lagrange's equations, Hamilton's equations, Boltzmann-Hamel equations, Gibbs equations, and Kane's equations. The most recent among these is Kane's method, based on a paper published in 1965 by Kane and Wang [1], and the method was given detailed exposition, with extensive applications, by Kane [2], Kane and Levinson [3], and Kane, Likins, Levinson [4]. Likins [5] also did a comparison of these various analytical methods for deriving equations of motion in a comprehensive report that also considered applications to flexible spacecraft.

In a survey paper, Kane and Levinson [6] took up a fairly complex example, of an 8 degree-of-freedom (dof) system consisting of a spacecraft containing a four-bar linkage to show the difference between seven analytical methods. To summarize, their conclusion was that (a) D'Alembert's method is less laborious than a method using conservation of momentum, with both requiring introduction and elimination of constraint forces; (b) Lagrange's equations require no introduction of workless constraint forces, but the labor to derive the equations is prohibitive; (c) Lagrange's equations in quasi-coordinates use variables that simplify the equations of motion but require order-n^3 computations for certain terms for an n-dof system, and the process of getting the final equations is formidable; (d) Gibbs equations is somewhat better, using quasi-coordinates but requiring one to form terms with n^2 computations for an n-dof system. With an exposition of Kane's method, they showed that Kane's method is superior to the rest of the methods, on the basis of two crucial considerations: (1) operational simplicity, meaning reduced labor in the derivation of the equations of motion either by hand or in terms of computer operations via symbol manipulation; and (2) simplicity of the final form of the equations, simplicity giving rise to reduction in computational time; simplicity is achievable depending on whether a method allows a choice of motion variables such as quasi-coordinates, or what Kane calls generalized speeds. An exposition of Kane's method is given later.

Flexible Multibody Dynamics: Efficient Formulations and Applications, First Edition. Arun K. Banerjee.

1.2 Kane's Method of Deriving Equations of Motion

Consider a system of particles and rigid bodies whose configuration in a Newtonian reference frame N is characterized by generalized coordinates, $q_1, q_2, \cdots, q_n$. Let $u_1, u_2, \ldots, u_n$ be motion variables, called *generalized speeds* by Kane, introduced as linear combinations of $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$, where an overdot indicates time derivative, that are kinematical differential equations of the form,

$$u_i = \sum_{j=1}^{n} W_{ij}\dot{q}_j + X_i \quad (i = 1, \ldots, n) \tag{1.1}$$

Here W_{ij} and X_i are functions of the generalized coordinates and time t, for an n-dof system. W_{ij} and X_i are chosen so that Eq. (1.1) can be uniquely solved for $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. Typically prescribed motion terms appear in X_i. The angular velocity of any rigid body and the velocity of any material point of the system can always be expressed uniquely as a linear function of the generalized speeds, $u_1, u_2, \ldots, u_n$. Thus, for a particle P_k in a system, Kane [3] has shown that its velocity in a Newtonian, meaning inertial, reference frame N, defined as the inertial time-derivative of the position vector of P_k from a point O fixed in N, can always be split in two groups of terms:

$$^N\mathbf{v}^{P_k} = \frac{^N d\mathbf{p}^{OP_k}}{dt} = \sum_{i=1}^{n} {^N\mathbf{v}_\mathrm{i}^{P_k}} u_\mathrm{i} + {^N\mathbf{v}_\mathrm{t}^{P_k}} \tag{1.2}$$

Here $^N\mathbf{v}_\mathrm{i}^{P_k}$, $^N\mathbf{v}_\mathrm{t}^{P_k}$ are vector functions of the generalized coordinates, $q_1, q_2, \ldots, q_n$. Kane [3] calls the vector, $^N\mathbf{v}_\mathrm{i}^{P_k}$, that is the coefficient of the *i*th generalized speed u_i in Eq. (1.2), the *i*th partial velocity of the point P_k. Similarly for a rigid body B_k, the velocity of its mass center B_k^* and the angular velocity of B_k in N for a system can always be expressed as

$$\begin{aligned} ^N\mathbf{v}^{B_k^*} &= \sum_{i=1}^{n} {^N\mathbf{v}_\mathrm{i}^{B_k^*}} u_i + {^N\mathbf{v}_\mathrm{t}^{B_k^*}} \\ ^N\boldsymbol{\omega}^{B_k} &= \sum_{i=1}^{n} {^N\boldsymbol{\omega}_\mathrm{i}^{B_k}} u_i + {^N\boldsymbol{\omega}_\mathrm{t}^{B_k}} \end{aligned} \tag{1.3}$$

Again, $^N\mathbf{v}_i^{B_k^*}$ is the *i*th partial velocity of B_k^*, and $^N\boldsymbol{\omega}_i^{B_k}, {^N\boldsymbol{\omega}_t^{B_k}}$ are vector functions of the generalized coordinates, and Kane calls the vector $^N\boldsymbol{\omega}_i^{B_k}$, the coefficient of u_i in $^N\boldsymbol{\omega}^{B_k}$ Eq. (1.3), the *i*th partial angular velocity of the body B_k in N. Typically, $^N\mathbf{v}_t^{P_k}, {^N\mathbf{v}_t^{B_k^*}}, {^N\boldsymbol{\omega}_t^{B_k}}$ in Eqs. (1.2) and (1.3) are remainder terms associated with prescribed velocity and angular velocity. Partial velocities and partial angular velocities are crucial items in Kane's method, and throughout this book we will see their central roles in the formulation of equations of motion. Once the velocities of points with mass and of mass centers and angular velocities of rigid bodies are expressed in some vector basis fixed in B_k, inertial acceleration of those points and mass centers, as well as angular acceleration of those bodies, can be obtained by differentiating these vector expressions in a Newtonian reference frame N. This is done by appealing to the rule for differentiation of a vector in two reference frames, expressed as:

$$
\begin{aligned}
{}^N\mathbf{a}^{P_k} &= \frac{{}^N d^N \mathbf{v}^{P_k}}{dt} = \frac{{}^{B_k} d^N \mathbf{v}^{P_k}}{dt} + {}^N\boldsymbol{\omega}^{B_k} \times {}^N\mathbf{v}^{P_k} \\
{}^N\mathbf{a}^{B_k^*} &= \frac{{}^N d^N \mathbf{v}^{B_k^*}}{dt} = \frac{{}^{B_k} d^N \mathbf{v}^{B_k^*}}{dt} + {}^N\boldsymbol{\omega}^{B_k} \times {}^N\mathbf{v}^{B_k^*} \\
{}^N\boldsymbol{\alpha}^{B_k} &= \frac{{}^N d^N \boldsymbol{\omega}^{B_k}}{dt} = \frac{{}^{B_k} d^N \boldsymbol{\omega}^{B_k}}{dt}
\end{aligned}
\tag{1.4}
$$

The first equality indicates a definition, and the second equality sign provides a basic kinematic relationship between differentiation of a vector in two reference frames, and it assumes that the frame B_k, or equivalently, rigid body B_k is different from frame N.

Kane's equations of motion are stated in terms of what Kane calls generalized inertia forces and generalized active forces. For an n-dof system consisting of NR number of rigid bodies and NP number of particles, the *i*th generalized inertia force is defined by the following dot-products with the *i*th partial velocities and partial angular velocities:

$$
\begin{aligned}
F_i^* = &- \sum_{j=1}^{NR} \left[m_j {}^N a^{B_j^*} \cdot {}^N\mathbf{v}_i^{B_j^*} + (\mathbf{I}^{B_j/B_j^*} \cdot {}^N\boldsymbol{\alpha}^{B_j} + {}^N\boldsymbol{\omega}^{B_j} \times \mathbf{I}^{B_j/B_j^*} \cdot {}^N\boldsymbol{\omega}^{B_j}) \cdot {}^N\boldsymbol{\omega}_i^{B_j} \right] \\
&- \sum_{j=1}^{NP} m_j {}^N\mathbf{a}^{P_j} \cdot {}^N\mathbf{v}_i^{P_j} \qquad (i = 1,\ldots\ldots, n)
\end{aligned}
\tag{1.5}
$$

Here ${}^N\mathbf{a}^{\mathbf{B}_j^*}$, ${}^N\mathbf{a}^{\mathbf{P}_j}$ are the Newtonian frame accelerations of the mass centers B_j^* of the body B_j and particle P_j, respectively; $\mathbf{I}^{B_j/B_j^*}$ is the inertia dyadic of B_j about B_j^*; and ${}^N\boldsymbol{\alpha}^{B_j}$ is the angular acceleration of B_j in N. The *i*th generalized active force for this n-dof system of NR number of rigid bodies and NP number of particles is given by the following dot-products with partial velocities and partial angular velocities:

$$
F_i = \sum_{j=1}^{NR} \left[\mathbf{F}^{\mathbf{B}_j^*} \cdot {}^N\mathbf{v}_i^{\mathbf{B}_j^*} + \mathbf{T}^{B_j} \cdot {}^N\boldsymbol{\omega}_i^{B_j}) \right] + \sum_{j=1}^{NP} \mathbf{F}^{\mathbf{P}_j} \cdot {}^N\mathbf{v}_i^{\mathbf{P}_j} \qquad (i = 1, \ldots, n) \tag{1.6}
$$

Here the resultant of all contact and body forces on body B_j are $\mathbf{F}^{B_j^*}$ at B_j^* together with a couple of torque $\mathbf{T}^{B_j}$, and the resultant of external and contact forces on particle P_j is $\mathbf{F}^{P_j}$. Note that all non-working interaction forces are automatically eliminated by taking the sum in Eq. (1.6) over bodies and particles, with actions and reactions canceling, as generalized active forces are formed. Some special cases of generalized active force that are covered by Eq. (1.6) are those due to elastic-dissipative mechanical systems, by "conservative" forces derivable from a potential function $V(q_1, \ldots, q_n, t)$ and dissipative forces from a dissipation function $D(u_1, \ldots, u_n)$.

$$F_i^c = -\frac{\partial V}{\partial q_i} - \frac{\partial D}{\partial u_i} \qquad (i = 1, \ldots, n) \tag{1.7}$$

1.2.1 Kane's Equations

Kane's Equations for an n-dof system can now be written by adding up the generalized active and inertia forces, and setting them equal to zero, as

$$F_i + F_i^* = 0, \quad i = 1, \ldots, n \quad \text{or,} \quad -F_i^* = F_i, \quad i = 1, \ldots, n \tag{1.8}$$

These dynamical equations of motion, together with the kinematical equation of Eq. (1.1) can be written as two sets of n coupled, nonlinear, differential equations in matrix form:

$$\begin{aligned} [M(q)]\{\dot{U}\} &= \{C(q, U, t)\} + \{F(q, U, t)\} \\ [W(q)]\{\dot{q}\} &= \{U - X(q, t)\} \end{aligned} \tag{1.9}$$

Here $M(q)$ is called the nxn "mass matrix," $C(q, U, t)$ the nx1 "Coriolis and centrifugal inertia force matrix," and $F(q, U, t)$ the nx1 "generalized force" matrix. Equation (1.9) completely describes the dynamics of the system. Note that the algebra involved in forming Eqs. (1.5) and (1.6) can be quite massive for a complex mechanical system, as may be checked by an analyst deriving equations of motion by hand. That is why a computerized symbol manipulation code, Autolev, was developed by Levinson and Kane [7] to derive the equations of motion. Finally, it should be mentioned that Kane had originally [2] called Eq. (1.8) the Lagrange's form of D'Alembert's Principle, because just as Lagrange's equations can be derived by dot-multiplying the D'Alembert force equilibrium equations by the components of virtual displacement in a virtual work principle, Kane obtains his equations by dot-multiplying the D'Alembert equilibrium equations by the partial velocities and partial angular velocities, to represent what may be thought of as a virtual power principle.

1.2.2 Simple Example: Equations for a Double Pendulum

Figure 1.1 shows a planar double pendulum. Consider the links OP, PQ as massless rigid rods, each of length l, with lumped mass m at the end of each rod acted on only by gravity. Configuration of the pendulum is defined by two generalized coordinates, q_1, q_2, as shown.

To use Kane's method we may choose as generalized speeds, following Eq. (1.1):

$$u_1 = \dot{q}_1; \quad u_2 = \dot{q}_1 + \dot{q}_2 \quad \text{or} \quad \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 - u_1 \end{Bmatrix} \tag{1.10}$$

The velocity of P in the Newtonian reference frame N can be written in terms of the angular velocity of the link OP in N, $u_1 \mathbf{n}_3$ ($\mathbf{n}_3$ being perpendicular to the plane in Figure 1.1) as

$${}^N\mathbf{v}^P = u_1 \mathbf{n}_3 \times l(\cos q_1 \mathbf{n}_1 + \sin q_1 \mathbf{n}_2) = l u_1(-\sin q_1 \mathbf{n}_1 + \cos q_1 \mathbf{n}_2) \tag{1.11}$$

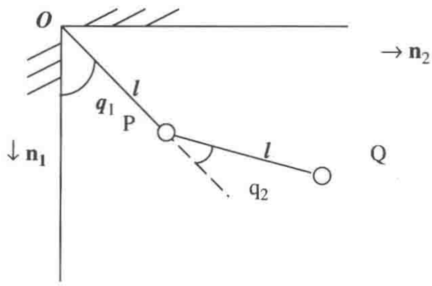

Figure 1.1 A Planar Double Pendulum.

Here $\mathbf{n}_1, \mathbf{n}_2$ are unit vectors in N directed downward and to the right, respectively, as shown in Figure 1.1. The velocity of Q in N is given in terms of the velocity of P in N by

$$\begin{aligned} {}^N\mathbf{v}^Q &= {}^N\mathbf{v}^P + u_2\mathbf{n}_3 \times l[(\cos(q_1+q_2)\,\mathbf{n}_1 + \sin(q_1+q_2)\,\mathbf{n}_2] \\ &= \mathbf{n}_1 l[-u_1 \sin q_1 - u_2 \sin(q_1+q_2)] + \mathbf{n}_2 l[u_1 \cos q_1 + u_2 \cos(q_1+q_2)] \end{aligned} \tag{1.12}$$

Here $u_2\mathbf{n}_3$ is the angular velocity in N of the link PQ. Now we form partial velocities of P and Q, coefficients of generalized speeds, u_1, u_2 in Eqs. (1.11), (1.12) shown in Table 1.1.

Accelerations of P and Q in N are obtained by differentiating the velocity vectors in N:

$${}^N\mathrm{a}^P = \frac{{}^N d^N\mathbf{v}^P}{dt} = l\left[\dot{u}_1(-\sin q_1 \mathrm{n}_1 + \cos q_1 \mathrm{n}_2) + u_1^2(-\cos q_1 \mathrm{n}_1 - \sin q_1 \mathrm{n}_2)\right] \tag{1.13}$$

$$\begin{aligned} {}^N\mathbf{a}^Q = \frac{{}^N d^N\mathbf{v}^Q}{dt} &= l\left[\dot{u}_1(-\sin q_1 \mathbf{n}_1 + \cos q_1 \mathbf{n}_2) + u_1^2(-\cos q_1 \mathbf{n}_1 - \sin q_1 \mathbf{n}_2)\right] \\ &+ l\left\{\dot{u}_2\left[-\sin(q_1+q_2)\,\mathbf{n}_1 + \cos(q_1+q_2)\,\mathbf{n}_2\right] + u_2^2[-\cos(q_1+q_2)\,\mathbf{n}_1 - \sin(q_1+q_2)\,\mathbf{n}_2]\right\} \end{aligned} \tag{1.14}$$

Negatives of the generalized inertia forces in two generalized speeds are formed by Eq. (1.5):

$$-F_1^* = ml^2\left[2\dot{u}_1 + \dot{u}_2 \cos q_2 - u_2^2 \sin q_2\right] \tag{1.15}$$

$$-F_2^* = ml^2\left[\dot{u}_1 \cos q_2 + u_1^2 \sin q_2 + \dot{u}_2\right] \tag{1.16}$$

These equations would be more complex had we chosen $u_i = \dot{q}_i, i = 1, 2$. Relative simplicity of the form of Eqs. (1.15), (1.16) is due to an efficient choice of generalized speeds, which

Table 1.1 Partial Velocities for the Double Pendulum Example.

r	${}^N\mathbf{v}_r^P$	${}^N\mathbf{v}_r^Q$
1	$l(-\sin q_1 \mathbf{n}_1 + \cos q_1 \mathbf{n}_2)$	$l(-\sin q_1 \mathbf{n}_1 + \cos q_1 \mathbf{n}_2)$
2	0	$l[-\sin(q_1+q_2)\,\mathbf{n}_1 + \cos(q_1+q_2)\,\mathbf{n}_2$

is described later in this chapter's appendix. The expressions for the generalized active forces due to gravity on P and Q are obtained via Eq. (1.6).

$$F_1 = \text{mg}\, \mathbf{n}_1 \bullet {}^N\mathbf{v}_1^P + mg\mathbf{n}_1 \cdot \bullet {}^N\mathbf{v}_1^Q = -2\,\text{mgl} \sin q_1 \tag{1.17}$$

$$F_2 = \text{mg}\mathbf{n}_1 \bullet {}^N\mathbf{v}_2^P + mg\mathbf{n}_1 \bullet {}^N\mathbf{v}_2^Q = -\text{mgl}\ \sin(q_1 + q_2) \tag{1.18}$$

Substitution of Eqs. (1.15)–(1.18) in Kane's equation, Eq. (1.8), yields the dynamical equations:

$$ml^2 \left[2\dot{u}_1 + \dot{u}_2 \cos q_2 - u_2^2 \sin q_2\right] = -2\,mgl \sin q_1 \tag{1.19}$$

$$ml^2 \left[\dot{u}_1 \cos q_2 + u_1^2 \sin q_2 + \dot{u}_2\right] = -mgl \sin(q_1 + q_2) \tag{1.20}$$

These dynamical equations are written in matrix form as:

$$\begin{bmatrix} 2ml^2 & ml^2 \cos q_2 \\ ml^2 \cos q_2 & ml^2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = (ml^2 \sin q_2) \begin{Bmatrix} u_2^2 \\ -u_1^2 \end{Bmatrix} - \begin{Bmatrix} 2mgl \sin q_1 \\ mgl \sin(q_1 + q_1) \end{Bmatrix} \tag{1.21}$$

Note that the multiplier matrix of the column matrix for the derivatives of the generalized speeds, the so-called mass matrix, is symmetric. This is true for all rigid body systems. Dynamical equations, Eq. (1.21), together with the kinematical equations, Eq. (1.10), complete the equations of motion for the double pendulum.

1.2.3 Equations for a Spinning Spacecraft with Three Rotors, Fuel Slosh, and Nutation Damper

Consider a more complex spacecraft example, for which equations of motion were derived by the author and reported in Ref. [8]. Figure 1.2 shows a spinning spacecraft with three-axis control, a nutation damper, and a thruster with thrust fuel sloshing in a tank. The spacecraft is a gyrostat G, meaning a rigid body with three fixed-axis reaction control rotors W_1, W_2, W_3; sloshing fuel is represented as a spherical pendulum with a massless rod with an end point mass m_P, and the nutation damper has a point mass m_Q, at point Q, with $\hat{Q}$ being the location in the spacecraft body G when the nutation damper spring is unstretched. The slosh pendulum attachment point is located from the mass center G* of G by the position vector $z_0\mathbf{g}_3$. We show in Figure 1.3 two angles Ψ_1, Ψ_2 to orient the slosh pendulum, and let σ denote the nominal length plus the elastic stretch of the spring at the nutation damper of mass m_Q. This describes a 12 dof system, for which we define the generalized speeds, u_i, $(i = 1, \ldots, 12)$, as follows:

$$u_i = {}^N\mathbf{v}^{G*} \cdot \mathbf{g}_i, \quad i = 1, 2, 3 \tag{1.22}$$

$$u_{3+i} = {}^N\boldsymbol{\omega}^G \cdot \mathbf{g}_i, \quad i = 1, 2, 3 \tag{1.23}$$

$$u_{6+i} = {}^G\boldsymbol{\omega}^{W_i} \cdot \mathbf{g}_i, \quad i = 1, 2, 3 \tag{1.24}$$

$$u_{9+i} = \dot{\psi}_i, \quad i = 1, 2 \tag{1.25}$$

$$u_{12} = \dot{\sigma} \tag{1.26}$$