CALCULUS OF SEVERAL VARIABLES

SECOND EDITION

Serge Lang

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FOREWORD

The present course on calculus of several variables is meant as a text, either for one semester following *A First Course in Calculus*, or for a year if the calculus sequence is so structured.

For a one-semester course, the first eight chapters provide an appropriate amount of material. If some time is left over, one can cover some topics in maxima-minima in Chapter XI, or the beginnings of higher derivatives and Taylor's formula in Chapter XII, depending on the taste of the instructor.

This first part has considerable unity of style. Many of the results are immediate corollaries of the chain rule. The main idea is that given a function of several variables, if we want to look at its values at two points P and Q, we join these points by a curve (often a straight line segment), and then look at the values of the function on that curve. By this device, we are able to reduce a large number of problems in several variables to problems and techniques in one variable. For instance, the tangent plane, the directional derivative, the law of conservation of energy, Taylor's formula, are all handled in this manner.

Green's theorem is more important to include in a one-semester course than other topics, because it provides a very elegant mixing of integration and differentiation techniques in one and two variables. This mixing is used frequently in applications, and also serves to fix these techniques in the mind because of the way they are used.

For a year's course, the rest of the book provides an adequate amount of material to be covered during the second semester. It consists of three topics, which are logically independent of each other and could be covered in any order. Some order must be chosen because it is necessary to project the course in a totally ordered way on the page axis (and the time axis), but logically, the choice is arbitrary. Pedagogically, the order chosen here seemed the one best suited for most people. These three topics are:

a) Triple integration and surface integrals, which continue ideas of Chapters VII and VIII.

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b) Maxima-minima and the Taylor formula, which continue the ideas of differentiating curves, perpendicularity, and analyzing a function of two or more variables by looking at its values on curves or line segments, thereby reducing the study of some properties to functions of one variable.

c) Matrices and determinants, which constitute the linear part of a function, and affect some properties like those of the inverse mapping theorem and the change of variables formula.

Different instructors will cover these three topics in whatever order they prefer. For applications to economics, it would make sense to cover the chapters on maxima-minima and the quadratic form in Taylor's formula before doing triple integration and surface integrals. The methods used depend only on the techniques developed as corollaries of the chain rule.

I think it is important that even at this early stage, students acquire the idea that one can operate with differentiation just as with polynomials. Thus §4 of Chapter X could be covered early, while leaving out entirely the much more theoretical section, §5, giving the proof of Taylor's formula in the general case. The proof is simple technically, but may cause some difficulty because it is a little abstract conceptually, although it does away with the usual mess of indices. (Just try to state the theorem without making use of the formalism of powers of derivatives!)

I have included only that part of linear algebra which is immediately useful for the applications to calculus. My *Introduction to Linear Algebra* provides an appropriate text when a whole semester is devoted to the subject. Many courses are still structured to give primary emphasis to the analytic aspects, and only a few notions involving matrices and linear maps are needed to cover, say, the chain rule for mappings of one space into another, and to emphasize the importance of linear approximations. These, it seems to me, are the essential ingredients of a second semester of calculus for students who want to become acquainted rapidly with the most important basic notions and how they are used in practice. Many years ago, there was no linear algebra introduced in calculus courses. Intermediate years have probably seen an excessive amount—more than was needed. I try to strike a proper balance here.

Some proofs have been included. On the whole, our policy has been to include those proofs which illustrate fundamental principles and are free of technicalities. Such proofs, which are also short, should be learned by students without difficulty. Examples are the uniqueness of the potential function, the law of conservation of energy, the independence of an integral on the path if a potential function exists, Green's theorem in the simplest cases, etc.

Other proofs, like those of the chain rule, or the local existence of a potential function, can be given in class or omitted, depending on the level of interest of a class and the taste of the instructor. For convenience, such proofs have usually been placed at the end of each section.

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Many worked-out examples have been added since previous editions, and answers to some exercises have been expanded to include more comprehensive solutions. I have done this to lighten the text on occasion. Such expanded solutions can also be viewed as worked-out examples simply placed differently, allowing students to think before they look up the answer if they have troubles with the problem.

I include two appendices on series and Fourier series, for the convenience of courses structured so that it is desirable to give an inkling of these topics some time during the second-year calculus, without waiting for a course in advanced calculus.

I would like to express my appreciation for the helpful guidance provided by the reviewers: M. B. Abrahamse (University of Virginia), Sherwood F. Ebey (University of the South), and William F. Keigher (The University of Tennessee).

I also thank Anthony Petrello for working out the answers and helping with the proofreading.

New Haven, Connecticut S. L.
January 1979

CONTENTS

CHAPTER I

Vectors

The concept of a vector is basic for the study of functions of several variables. It provides geometric motivation for everything that follows. Hence the properties of vectors, both algebraic and geometric, will be discussed in full.

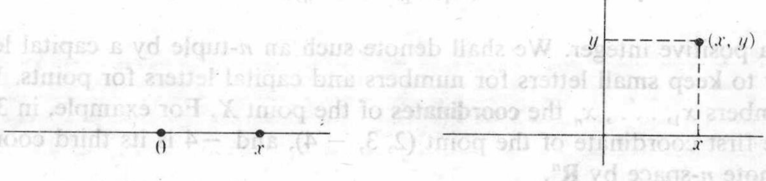
One significant feature of all the statements and proofs of this part is that they are neither easier nor harder to prove in 3-space than they are in 2-space. Since we have to deal with $n = 2$ and $n = 3$, it is just as easy to state some things just with a neutral n. Also for physics and economics, it is useful to get used to n rather than 2 or 3. However, for purposes of pedagogy, throughout the book we always give first the definitions and formulas for the special cases of $n = 2$ and $n = 3$ so that the reader can omit any reference to higher n if he wishes.

§1. DEFINITION OF POINTS IN SPACE

We know that a number can be used to represent a point on a line, once a unit length is selected.

A pair of numbers (i.e. a couple of numbers) (x, y) can be used to represent a point in the plane.

These can be pictured as follows:



(a) Point on a line (b) Point in a plane

Figure 1

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We now observe that a triple of numbers (x, y, z) can be used to represent a point in space, that is 3-dimensional space, or 3-space. We simply introduce one more axis.

Figure 2 illustrates this.

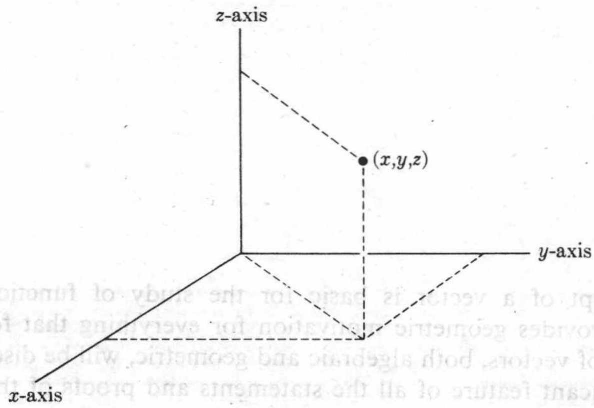

Figure 2

Instead of using x, y, z we could also use (x_1, x_2, x_3). The line could be called 1-space, and the plane could be called 2-space.

Thus we can say that a single number represents a point in 1-space. A couple represents a point in 2-space. A triple represents a point in 3-space.

Although we cannot draw a picture to go further, there is nothing to prevent us from considering a quadruple of numbers

$$(x_1, x_2, x_3, x_4)$$

and decreeing that this is a point in 4-space. A quintuple would be a point in 5-space, then would come a sextuple, septuple, octuple, ...

We let ourselves be carried away and **define a point in n-space** to be an n-tuple of numbers

$$(x_1, x_2, \ldots, x_n),$$

if n is a positive integer. We shall denote such an n-tuple by a capital letter X, and try to keep small letters for numbers and capital letters for points. We call the numbers $x_1, \ldots, x_n$ the **coordinates** of the point X. For example, in 3-space, 2 is the first coordinate of the point $(2, 3, -4)$, and -4 is its third coordinate. We denote n-space by $\mathbf{R}^n$.

Most of our examples will take place when $n = 2$ or $n = 3$. Thus the reader may visualize either of these two cases throughout the book. However, three comments must be made.

First, we have to handle $n = 2$ and $n = 3$, so that in order to avoid a lot of repetitions, it is useful to have a notation which covers both these cases simultaneously, even if we often repeat the formulation of certain results separately for both cases.

Second, no theorem or formula is simpler by making the assumption that $n = 2$ or 3.

Third, the case $n = 4$ does occur in physics.

Example 1. One classical example of 3-space is of course the space we live in. After we have selected an origin and a coordinate system, we can describe the position of a point (body, particle, etc.) by 3 coordinates. Furthermore, as was known long ago, it is convenient to extend this space to a 4-dimensional space, with the fourth coordinate as time, the time origin being selected, say, as the birth of Christ—although this is purely arbitrary (it might be more convenient to select the birth of the solar system, or the birth of the earth as the origin, if we could determine these accurately). Then a point with negative time coordinate is a BC point, and a point with positive time coordinate is an AD point.

Don't get the idea that "time is *the* fourth dimension," however. The above 4-dimensional space is only one possible example. In economics, for instance, one uses a very different space, taking for coordinates, say, the number of dollars expended in an industry. For instance, we could deal with a 7-dimensional space with coordinates corresponding to the following industries:

1. Steel	2. Auto	3. Farm products	4. Fish
5. Chemicals	6. Clothing	7. Transportation	

We agree that a megabuck per year is the unit of measurement. Then a point

$$(1{,}000, 800, 550, 300, 700, 200, 900)$$

in this 7-space would mean that the steel industry spent one billion dollars in the given year, and that the chemical industry spent 700 million dollars in that year.

We shall now define how to add points. If A, B are two points, say in 3-space,

$$A = (a_1, a_2, a_3) \qquad \text{and} \qquad B = (b_1, b_2, b_3)$$

then we define $A + B$ to be the point whose coordinates are

$$A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

Example 2. In the plane, if $A = (1, 2)$ and $B = (-3, 5)$, then

$$A + B = (-2, 7).$$

In 3-space, if $A = (-1, \pi, 3)$ and $B = (\sqrt{2}, 7, -2)$, then

$$A + B = (\sqrt{2} - 1, \pi + 7, 1).$$

Using a neutral n to cover both the cases of 2-space and 3-space, the points would be written

$$A = (a_1, \ldots, a_n), \qquad B = (b_1, \ldots, b_n),$$

and we define $A + B$ to be the point whose coordinates are

$$(a_1 + b_1, \ldots, a_n + b_n).$$

We observe that the following rules are satisfied:

1. $(A + B) + C = A + (B + C)$.
2. $A + B = B + A$.
3. If we let

$$O = (0, 0, \ldots, 0)$$

be the point all of whose coordinates are 0, then

$$O + A = A + O = A$$

for all A.

4. Let $A = (a_1, \ldots, a_n)$ and let $-A = (-a_1, \ldots, -a_n)$. Then

$$A + (-A) = O.$$

All these properties are very simple, and are true because they are true for numbers, and addition of n-tuples is defined in terms of addition of their components, which are numbers.

Note. Do not confuse the number 0 and the n-tuple $(0, \ldots, 0)$. We usually denote this n-tuple by O, and also call it zero, because no difficulty can occur in practice.

We shall now interpret addition and multiplication by numbers geometrically in the plane (you can visualize simultaneously what happens in 3-space).

Example 3. Let $A = (2, 3)$ and $B = (-1, 1)$. Then

$$A + B = (1, 4).$$

The figure looks like a **parallelogram** (Fig. 3).

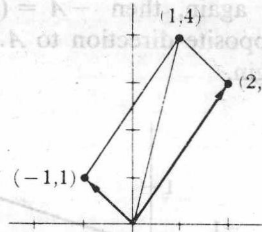

Figure 3

Example 4. Let $A = (3, 1)$ and $B = (1, 2)$. Then

$$A + B = (4, 3).$$

We see again that the geometric representation of our addition looks like a **parallelogram** (Fig. 4).

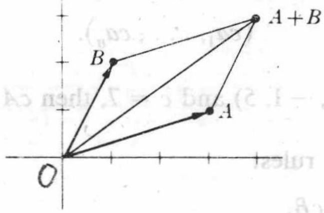

Figure 4

The reason why the figure looks like a **parallelogram** can be given in terms of plane geometry as follows. We obtain $B = (1, 2)$ by starting from the origin $O = (0, 0)$, and moving 1 unit to the right and 2 up. To get $A + B$, we start from A, and again move 1 unit to the right and 2 up. Thus the line segments between O and B, and between A and $A + B$ are the hypotenuses of right triangles whose corresponding legs are of the same length, and parallel. The above segments are therefore parallel and of the same length, as illustrated in Fig. 5.

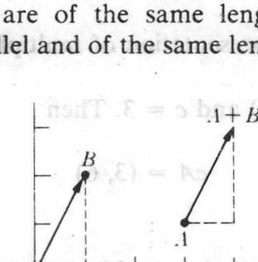

Figure 5

Example 5. If $A = (3, 1)$ again, then $-A = (-3, -1)$. If we plot this point, we see that $-A$ has opposite direction to A. We may view $-A$ as the reflection of A through the origin.

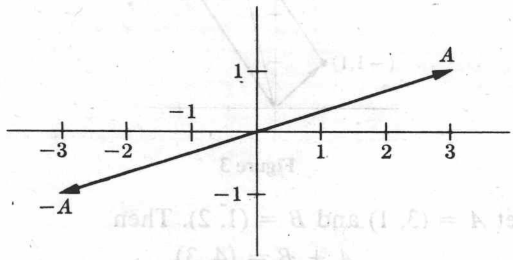

Figure 6

We shall now consider multiplication of A by a number. If c is any number, we **define** cA to be the point whose coordinates are

$$(ca_1, \ldots; ca_n).$$

Example 6. If $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$.

It is easy to verify the rules:

5. $c(A + B) = cA + cB$.
6. If c_1, c_2 are numbers, then

$$(c_1 + c_2)A = c_1A + c_2A \quad \text{and} \quad (c_1c_2)A = c_1(c_2A).$$

Also note that

$$(-1)A = -A.$$

What is the geometric representation of multiplication by a number?

Example 7. Let $A = (1, 2)$ and $c = 3$. Then

$$cA = (3, 6)$$

as in Fig. 7 (a).

Multiplication by 3 amounts to stretching A by 3. Similarly, $\frac{1}{2}A$ amounts to stretching A by $\frac{1}{2}$, i.e. shrinking A to half its size. In general, if t is a number,

$t > 0$, we interpret tA as a point in the same direction as A from the origin, but t times the distance. In fact, we define A and B to have the **same direction** if there exists a number $c > 0$ such that $A = cB$. We emphasize that this means A and B have the same direction **with respect to the origin**. For simplicity of language, we omit the words "with respect to the origin."

Multiplication by a negative number reverses the direction. Thus $-3A$ would be represented as in Fig. 7 (b).

We define two vectors A, B (neither of which is zero) to have **opposite directions** if there is a number $c < 0$ such that $cA = B$. Thus when $B = -A$, then A, B have opposite direction.

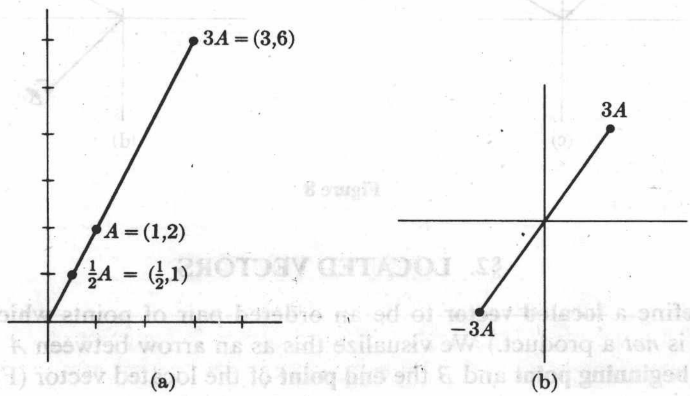

Figure 7

EXERCISES

Find $A + B, A - B, 3A, -2B$ in each of the following cases. Draw the points of Exercises 1 and 2 on a sheet of graph paper.

1. $A = (2, -1), B = (-1, 1)$
2. $A = (-1, 3), B = (0, 4)$
3. $A = (2, -1, 5), B = (-1, 1, 1)$
4. $A = (-1, -2, 3), B = (-1, 3, -4)$
5. $A = (\pi, 3, -1), B = (2\pi, -3, 7)$
6. $A = (15, -2, 4), B = (\pi, 3, -1)$
7. Let $A = (1, 2)$ and $B = (3, 1)$. Draw $A + B, A + 2B, A + 3B, A - B, A - 2B, A - 3B$ on a sheet of graph paper.
8. Let A, B be as in Exercise 1. Draw the points $A + 2B, A + 3B, A - 2B, A - 3B, A + \frac{1}{2}B$ on a sheet of graph paper.
9. Let A and B be as drawn in Fig. 8. Draw the point $A - B$.

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