# CALCULUS OF SEVERAL VARIABLES SECOND EDITION

Serge Lang

51.612 L27/

# CALCULUS OF SEVERAL VARIABLES

# SERGE LANG

Yale University

SECOND EDITION

## ADDISON-WESLEY PUBLISHING COMPANY

Reading, Massachusetts · Menlo Park, California London · Amsterdam · Don Mills, Ontario · Sydney

# This book is in the ADDISON-WESLEY SERIES IN MATHEMATICS

## LYNN H. LOOMIS Consulting Editor

### Library of Congress Cataloging in Publication Data

Lang, Serge, 1927-Calculus of several variables.

Includes index.
1. Calculus. 2. Functions of several real variables. I. Title.
QA303.1256 1979 515'.84 78-55823
ISBN 0-201-04299-1

Copyright © 1979, 1973, 1968 by Addison-Wesley Publishing Company, Inc. Philippines copyright 1979, 1973 by Addison-Wesley Publishing Company, Inc.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher. Printed in the United States of America. Published simultaneously in Canada. Library of Congress Catalog Card No. 78-55823.

ISBN 0-201-04299-1 ABCDEFGHIJK-MA-79

# **FOREWORD**

The present course on calculus of several variables is meant as a text, either for one semester following A First Course in Calculus, or for a year if the calculus sequence is so structured.

For a one-semester course, the first eight chapters provide an appropriate amount of material. If some time is left over, one can cover some topics in maxima-minima in Chapter XI, or the beginnings of higher derivatives and Taylor's formula in Chapter XII, depending on the taste of the instructor.

This first part has considerable unity of style. Many of the results are immediate corollaries of the chain rule. The main idea is that given a function of several variables, if we want to look at its values at two points P and Q, we join these points by a curve (often a straight line segment), and then look at the values of the function on that curve. By this device, we are able to reduce a large number of problems in several variables to problems and techniques in one variable. For instance, the tangent plane, the directional derivative, the law of conservation of energy, Taylor's formula, are all handled in this manner.

Green's theorem is more important to include in a one-semester course than other topics, because it provides a very elegant mixing of integration and differentiation techniques in one and two variables. This mixing is used frequently in applications, and also serves to fix these techniques in the mind because of the way they are used.

For a year's course, the rest of the book provides an adequate amount of material to be covered during the second semester. It consists of three topics, which are logically independent of each other and could be covered in any order. Some order must be chosen because it is necessary to project the course in a totally ordered way on the page axis (and the time axis), but logically, the choice is arbitrary. Pedagogically, the order chosen here seemed the one best suited for most people. These three topics are:

a) Triple integration and surface integrals, which continue ideas of Chapters VII and VIII.

- b) Maxima-minima and the Taylor formula, which continue the ideas of differentiating curves, perpendicularity, and analyzing a function of two or more variables by looking at its values on curves or line segments, thereby reducing the study of some properties to functions of one variable.
- c) Matrices and determinants, which constitute the linear part of a function, and affect some properties like those of the inverse mapping theorem and the change of variables formula.

Different instructors will cover these three topics in whatever order they prefer. For applications to economics, it would make sense to cover the chapters on maxima-minima and the quadratic form in Taylor's formula before doing triple integration and surface integrals. The methods used depend only on the techniques developed as corollaries of the chain rule.

I think it is important that even at this early stage, students acquire the idea that one can operate with differentiation just as with polynomials. Thus §4 of Chapter X could be covered early, while leaving out entirely the much more theoretical section, §5, giving the proof of Taylor's formula in the general case. The proof is simple technically, but may cause some difficulty because it is a little abstract conceptually, although it does away with the usual mess of indices. (Just try to state the theorem without making use of the formalism of powers of derivatives!)

I have included only that part of linear algebra which is immediately useful for the applications to calculus. My *Introduction to Linear Algebra* provides an appropriate text when a whole semester is devoted to the subject. Many courses are still structured to give primary emphasis to the analytic aspects, and only a few notions involving matrices and linear maps are needed to cover, say, the chain rule for mappings of one space into another, and to emphasize the importance of linear approximations. These, it seems to me, are the essential ingredients of a second semester of calculus for students who want to become acquainted rapidly with the most important basic notions and how they are used in practice. Many years ago, there was no linear algebra introduced in calculus courses. Intermediate years have probably seen an excessive amount—more than was needed. I try to strike a proper balance here.

Some proofs have been included. On the whole, our policy has been to include those proofs which illustrate fundamental principles and are free of technicalities. Such proofs, which are also short, should be learned by students without difficulty. Examples are the uniqueness of the potential function, the law of conservation of energy, the independence of an integral on the path if a potential function exists, Green's theorem in the simplest cases, etc.

Other proofs, like those of the chain rule, or the local existence of a potential function, can be given in class or omitted, depending on the level of interest of a class and the taste of the instructor. For convenience, such proofs have usually been placed at the end of each section.

整PDF语访问: www.ertongbook.com

527720

Many worked-out examples have been added since previous editions, and answers to some exercises have been expanded to include more comprehensive solutions. I have done this to lighten the text on occasion. Such expanded solutions can also be viewed as worked-out examples simply placed differently, allowing students to think before they look up the answer if they have troubles with the problem.

I include two appendices on series and Fourier series, for the convenience of courses structured so that it is desirable to give an inkling of these topics some time during the second-year calculus, without waiting for a course in advanced calculus.

I would like to express my appreciation for the helpful guidance provided by the reviewers: M. B. Abrahamse (University of Virginia), Sherwood F. Ebey (University of the South), and William F. Keigher (The University of Tennessee).

I also thank Anthony Petrello for working out the answers and helping with the proofreading.

New Haven, Connecticut January 1979 S. L.

# CONTENTS

| 177         |  |                                     | SIND TROOP BIR TO TOUR Y                       |           |      |
|-------------|--|-------------------------------------|--|-----------|------|
|             |  |                                     | Curve lutegrais                                | IV tokpa  | D.   |
|             | Definition and evaluation of curve integrals 127 |                                     |  |           |      |
|             |  |                                     | The reverse path                               |           |      |
|             |  | the vector field has a potential fu |  |           |      |
|             |  | legral on the path                  | Dependence of the in                           | þ.        |      |
| Chap        | ter I  | Vectors                             | Dooble lategrals                               | IIV votqs |      |
| 151         | 1  | Definition of points in space       | Dottble integrals                              |           | 1    |
|             | 2  | Located vectors                     |  |           | 8    |
|             | 3  | Scalar product                      | Polar coordinates                              |           | 11   |
|             | 4  | The norm of a vector                |  |           | 14   |
|             | -5   | Parametric lines                    | Green's Theorem                                | UIV rein  | 25   |
|             | -  | Planes                              | and the second                                 |           | 29   |
| 179         | 7  | The cross product                   | The standard version<br>The divergence and the | 2         | 35   |
| Chapter II  |  | Differentiation of Vectors          | Triple Integrals                               | XI tëtpa  | Ó    |
|             | 1  | Derivative                          |  |           | 41   |
| 199         | - 2  | Length of curves                    | Triple integrals                               |           | 52   |
| 203         |  | cal coordinates                     | Cylindrical and spheri                         |           |      |
| Chapter III |  | Functions of Several Variables      | Center of mass                                 |           |      |
|             | 1  | Graphs and level curves             | Security of the second                         |           | - 55 |
|             | 2  | Partial derivatives                 | Surface Integrals                              |           | 60   |
| 219         | 3  | Differentiability and gradient      | Parametrization, tange                         |           | 66   |
|             | 4  | Repeated partial derivatives        | Surface area                                   | 2 77      | 70   |
|             |  |                                     | Surface integrals                              | 2         |      |
| Chapte      | er IV  | The Chain Rule and the Gradient     | tiurl and divergence of                        | Þ.        |      |
| 243         | 1  | The chain rule                      | Divergence theorem in                          | É         | 75   |
| 252         | 2  | Tangent plane                       | Stokes' theorem                                | 9         | 80   |
|             | . 3  | Directional derivative              |  |           | 85   |
|             | 4  | Functions depending only on the     | Maximum and Mining                             | Spiter XI |      |
|             |  | distance from the origin            | Critical points                                |           | 89   |
| 107         | 5  | Further technique in partial diffe  | erentiation                                    |           | 97   |
| 264         |  |                                     | weilmillin nonema l                            |           |      |

| Chapter V                                | Potential Functions  |     |  |
|--|--|-----|--|
| 1  | Conservation law   | 103 |  |
| 2  | Potential functions  | 106 |  |
| 3 Local existence of potential functions |  |     |  |
| 4 An important special vector field      |  |     |  |
|  | Differentiating under the integral                             | 119 |  |
|  | Proof of the local existence theorem                           | 121 |  |
| Chapter VI                               | Curve Integrals  |     |  |
| 1  | Definition and evaluation of curve integrals                   | 127 |  |
|  | The reverse path   |     |  |
|  | Curve integrals when the vector field has a potential function | 138 |  |
|  | Dependence of the integral on the path                         | 146 |  |
| Chapter VII                              | Double Integrals   | Cha |  |
| 1  | Double integrals space in space of points in space             | 151 |  |
|  | Repeated integrals   | 158 |  |
|  | Polar coordinates  | 168 |  |
|  | 4 The norm of a vector   | 1   |  |
| Chanter VIII                             | Green's Theorem  |     |  |
| Chapter yill                             | diens incorem  |     |  |
|  | The standard version   | 179 |  |
| 2  | The divergence and the rotation of a vector field              | 189 |  |
|  | ter II. Differentiation of Vectors                             |     |  |
| Chapter IX                               | Triple Integrals   |     |  |
| 1  | Triple integrals   | 199 |  |
|  | Cylindrical and spherical coordinates                          | 203 |  |
|  | Center of mass   | 215 |  |
| 200                                      | l' Graphs and level curves                                     |     |  |
| 1111                                     | Surface Integrals  |     |  |
| 88 1                                     | Parametrization, tangent plane, and normal vector              | 219 |  |
|  | Surface area Surface area Repeated partial derivatives         | 225 |  |
|  | Surface integrals  | 232 |  |
| . 4                                      | Curl and divergence of a vector field a sluff mind still vitas | 241 |  |
| 5  | Divergence theorem in 3-space                                  | 243 |  |
| 6  | Stokes' theorem  | 252 |  |
| 0.8                                      | Directional derivative   |     |  |
| Chapter X                                | Maximum and Minimum no vitro antibrage beautions if            |     |  |
| 98 1                                     | Critical points interest of the most sensitivity               | 261 |  |
| 70 2                                     | Boundary points the state of the superinteer red to 1          | 264 |  |
| 3  | Lagrange multipliers   | 269 |  |

| Chapter XII  | Higher Derivatives   | Appendix |
|--------------|--|----------|
| 1 407        | The first two terms in Taylor's formula   SOZ   Internal                     | 277      |
| 2 413        | The quadratic term at critical points o moust agree 3                        | 281      |
|              | Algebraic study of a quadratic form  | 287      |
| 4            | Partial differential operators   | 292      |
| 124. 5       | The general expression for Taylor's formula                                  | 298      |
| Chapter XIII | Matrices With positive terms 2. Series with positive terms 3. The ratio test |          |
| 429          | Matrices Designation of A  | 305      |
|              | Multiplication of matrices galacter and allernating Solution of Matrices     | 309      |
| Chapter XIV  | Linear Mappings to not arrest and not desired to                             |          |
| 1            | Mappings   | 317      |
|              | Linear mappings  | 323      |
|              | Geometric applications   | 328      |
| . 4          | Composition and inverse of mappings  | 334      |
| Chapter XV   | Determinants   |          |
| 1            | Determinants of order 2  | 341      |
| 2            | Determinants of order 3  | 345      |
| 3            | Additional properties of determinants  | 349      |
| 4            | Independence of vectors  | 356      |
| 5            | Determinant of a product.  | 358      |
| 6            | Inverse of a matrix  | 359      |
| Chapter XVI  | Applications to Functions of Several Variables                               |          |
| - 1          | The Jacobian matrix  | 361      |
| 2            | Differentiability  | 365      |
| 3            | The chain rule   | 367      |
| 4            | Inverse mappings   | 369      |
| 5            | Implicit functions   | 372      |
| 6            | The Hessian  | 376      |
| Chapter XVII | The Change of Variables Formula  |          |
| . 1          | Determinants as area and volume  | 379      |
|              | Dilations  | 387      |
| 3            | Change of variables formula in two dimensions                                | 392      |
|              | Application of Green's formula to the change of                              |          |
|              | variables formula  | 396      |
| 5            | Change of variables formula in three dimensions                              | 399      |
| 6            | Vector fields on the sphere  | 403      |

xii

### CONTENTS

| Appendix | 1 | Fourier Series eavituring radgitt                                    | Chapter XII  |
|----------|---|--|--------------|
| 277      | 1 | General scalar products was mission to the first sense own sense of  | 407          |
| 182      | 2 |  | 413          |
| 287      |   | Algebraic study of a quadratic form                                  |              |
| Appendix | 2 | Partial differential operators                                       |              |
| DXX.     | 1 | Convergent series  | 421          |
|          | 2 | Series with positive terms   | 424          |
|          |   | The ratio test   | 421          |
|          |   | The integral test  | 429          |
|          |   | Absolute and alternating convergence opening but M                   | 432          |
|          | 7 | Differentiation and integration of power series                      |              |
|          | 1 | Differentiation and integration of power series                      | 438          |
| Answers  |   | Mappings   | 443          |
| 575      |   | Linear mappings  |              |
| Index    |   | Geometric applications   | 477          |
|          |   | Composition and inverse of mappings                                  | 4            |
|          |   | Determinants   | Claspter XV  |
|          |   |  | YA DRIBBIO   |
| 1AE '    |   | Determinants of order 2  | 1            |
| 345      |   | Determinants of order 3  |              |
| 356      |   | Additional properties of determinants<br>Independence of vectors     | 3            |
|          |   | Determinant of a product   | 2            |
|          |   | Inverse of a matrix  |              |
|          |   |  |              |
|          |   | Applications to Functions of Several Variables                       | Chapter XVI  |
| 361      |   | The Jacobian matrix  | I -          |
| 365      |   | Differentiability  | 4 7 7 8      |
| 367      |   |  |              |
| 369      |   | Inverse mappings   |              |
|          |   | Implicat functions   |              |
|          |   | The Hessian  |              |
|          |   | The Change of Variables Formula                                      | Chapter XVII |
|          |   |  |              |
| 379      |   | Determinants as area and volume                                      |              |
|          |   | Dilations  |              |
|          |   | Change of variables formula in two dimensions                        |              |
| 396      |   | Application of Green's formula to the change of<br>variables formula |              |
| 996      |   | variables formula.  Change of variables formula in three dimensions. |              |
| COL      |   | CHOICIGHTO AND IT BIDDING COMMINE TO AGREEM                          |              |

# We now observe the RAPTER to represent a

# Vectors

The concept of a vector is basic for the study of functions of several variables. It provides geometric motivation for everything that follows. Hence the properties of vectors, both algebraic and geometric, will be discussed in full.

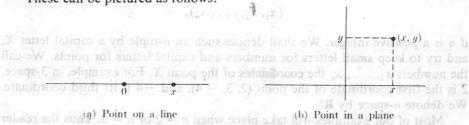
One significant feature of all the statements and proofs of this part is that they are neither easier nor harder to prove in 3-space than they are in 2-space. Since we have to deal with n = 2 and n = 3, it is just as easy to state some things just with a neutral n. Also for physics and economics, it is useful to get used to n rather than 2 or 3. However, for purposes of pedagogy, throughout the book we always give first the definitions and formulas for the special cases of n=2 and n=3 so that the reader can omit any reference to higher n if he Although we cannot draw a picture to go further, there is nothing to leading

### §1. DEFINITION OF POINTS IN SPACE

We know that a number can be used to represent a point on a line, once a unit length is selected.

A pair of numbers (i.e. a couple of numbers) (x, y) can be used to represent a point in the plane.

These can be pictured as follows:



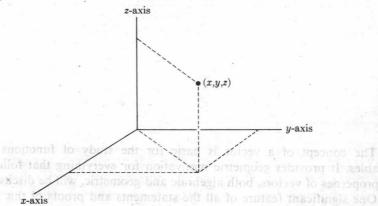
comments must be made.

Figure 1

雲更完整PDF请访问: www.ertongbook.com。

We now observe that a triple of numbers (x, y, z) can be used to represent a point in space, that is 3-dimensional space, or 3-space. We simply introduce one more axis.

Figure 2 illustrates this.



they are neither easier not harder to 2 spuge? In 3-space than they are in 2-space.

we have too deal with a = Instead of using x, y, z we could also use  $(x_1, x_2, x_3)$ . The line could be called 1-space, and the plane could be called 2-space.

Thus we can say that a single number represents a point in 1-space. A couple represents a point in 2-space. A triple represents a point in 3-space.

Although we cannot draw a picture to go further, there is nothing to prevent us from considering a quadruple of numbers

$$(x_1, x_2, x_3, x_4)$$

and decreeing that this is a point in 4-space. A quintuple would be a point in 5-space, then would come a sextuple, septuple, octuple, . . . .

We let ourselves be carried away and define a point in n-space to be an *n*-tuple of numbers ( $x_1, x_2, \ldots, x_n$ ),

$$(x_1, x_2, \ldots, x_n),$$

if n is a positive integer. We shall denote such an n-tuple by a capital letter X, and try to keep small letters for numbers and capital letters for points. We call the numbers  $x_1, \ldots, x_n$  the coordinates of the point X. For example, in 3-space, 2 is the first coordinate of the point (2, 3, -4), and -4 is its third coordinate. We denote *n*-space by  $\mathbb{R}^n$ .

Most of our examples will take place when n = 2 or n = 3. Thus the reader may visualize either of these two cases throughout the book. However, three comments must be made.

First, we have to handle n = 2 and n = 3, so that in order to avoid a lot of repetitions, it is useful to have a notation which covers both these cases simultaneously, even if we often repeat the formulation of certain results separately for both cases.

Second, no theorem or formula is simpler by making the assumption that n = 2 or 3.

Third, the case n = 4 does occur in physics.

Example 1. One classical example of 3-space is of course the space we live in. After we have selected an origin and a coordinate system, we can describe the position of a point (body, particle, etc.) by 3 coordinates. Furthermore, as was known long ago, it is convenient to extend this space to a 4-dimensional space, with the fourth coordinate as time, the time origin being selected, say, as the birth of Christ—although this is purely arbitrary (it might be more convenient to select the birth of the solar system, or the birth of the earth as the origin, if we could determine these accurately). Then a point with negative time coordinate is a BC point, and a point with positive time coordinate is an AD point.

Don't get the idea that "time is *the* fourth dimension," however. The above 4-dimensional space is only one possible example. In economics, for instance, one uses a very different space, taking for coordinates, say, the number of dollars expended in an industry. For instance, we could deal with a 7-dimensional space with coordinates corresponding to the following industries:

1. Steel

2. Auto

3. Farm products 4. Fish

5. Chemicals

6. Clothing 7. Transportation

We agree that a megabuck per year is the unit of measurement. Then a point

in this 7-space would mean that the steel industry spent one billion dollars in the given year, and that the chemical industry spent 700 million dollars in that year.

We shall now define how to add points. If A, B are two points, say in 3-space,

$$A = (a_1, a_2, a_3)$$
 and  $B = (b_1, b_2, b_3)$ 

then we define A + B to be the point whose coordinates are

$$A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

**Example 2.** In the plane, if A = (1, 2) and B = (-3, 5), then

$$A + B = (-2, 7).$$

In 3-space, if  $A = (-1, \pi, 3)$  and  $B = (\sqrt{2}, 7, -2)$ , then

$$A + B = (\sqrt{2} - 1, \pi + 7, 1).$$

Using a neutral n to cover both the cases of 2-space and 3-space, the points would be written  $\frac{1}{2}$  and  $\frac{1}{2}$  are the points

simultaneously even if we often repeat the formulation of certain results separately for both 
$$A = (a_1, \ldots, a_n), B = (b_1, \ldots, b_n),$$

Second, no theorem of formula is simpler by making the assumption that and we define A + B to be the point whose coordinates are

Example 1. One classic 
$$(a_1 + b_1, \dots, a_n + b_1)$$
 of course the space we live

Third, the case n = 4 does occur in physics

of the we have selected an origin and a coordinate system, we can describe the following rules are satisfied: (b.: Boart de position of a point (b.: Bartisfied) as

1. 
$$(A + B) + C = A + (B + C)$$
. In the second seco

space, with the Aouria coordinate as quie, the difference 
$$A+B=B+A\cdot C$$
 onto the birth of Christ-although this is purely arbitrary  $A+B+B+C$  on

venient to select the birth of the solar system, or the birth of 18 William 18.

origin, if we could determine these accurately). Then a point with negative time coordinate is a BC point, 
$$(0, \dots, 0, 0) \equiv 0$$
 sative time coordinate is an AD

be the point all of whose coordinates are 0, then sold and long and

one uses a very different 
$$A = 0 + A = A + 0$$
 ordinates, say, the number of

dollars expended in an industry. For instance, we could deal with a 7-dimen-

for all 
$$A$$
.

4. Let  $A = (a_1, \ldots, a_n)$  and let  $-A = (-a_1, \ldots, -a_n)$ . Then

5. Chemicals 
$$\int_{0}^{\infty} O(h) dh = \int_{0}^{\infty} A^{T} h$$
 Transportation We agree that a megabuck per year is the unit of measurement. Then a point

All these properties are very simple, and are true because they are true for numbers, and addition of *n*-tuples is defined in terms of addition of their components, which are numbers.

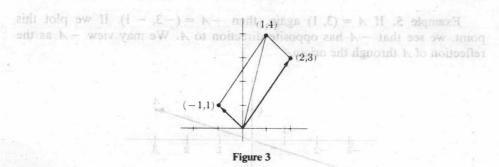
**Note.** Do not confuse the number 0 and the n-tuple  $(0, \ldots, 0)$ . We usually denote this n-tuple by O, and also call it zero, because no difficulty can occur in practice.

We shall now interpret addition and multiplication by numbers geometrically in the plane (you can visualize simultaneously what happens in 3-space).

**Example 3.** Let A = (2, 3) and B = (-1, 1). Then

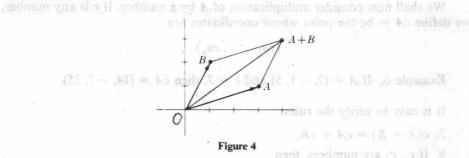
$$A + B = (1, 4).$$

The figure looks like a parallelogram (Fig. 3).



**Example 4.** Let A = (3, 1) and B = (1, 2). Then A + B = (4, 3).

We see again that the geometric representation of our addition looks like a parallelogram (Fig. 4).



The reason why the figure looks like a **parallelogram** can be given in terms of plane geometry as follows. We obtain B = (1, 2) by starting from the origin O = (0, 0), and moving 1 unit to the right and 2 up. To get A + B, we start from A, and again move 1 unit to the right and 2 up. Thus the line segments between A and A + B are the hypotenuses of right triangles whose corresponding legs are of the same length, and parallel. The above segments are therefore parallel and of the same length, as illustrated in Fig. 5.

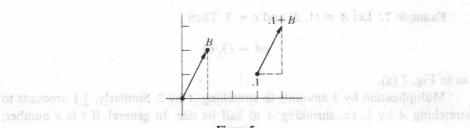
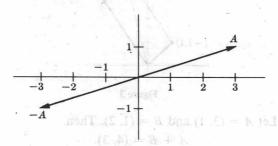


Figure 5

**Example 5.** If A = (3, 1) again, then -A = (-3, -1). If we plot this point, we see that -A has opposite direction to A. We may view -A as the reflection of A through the origin.



We see again that the geometric ro anugitation of our addition looks like a

We shall now consider multiplication of A by a number. If c is any number, we define cA to be the point whose coordinates are

$$(ca_1,\ldots,ca_n).$$

**Example 6.** If A = (2, -1, 5) and c = 7, then cA = (14, -7, 35).

It is easy to verify the rules:

- 5. c(A+B) = cA + cB.
- 6. If  $c_1$ ,  $c_2$  are numbers, then

$$(c_1 + c_2)A = c_1A + c_2A$$
 and  $(c_1c_2)A = c_1(c_2A)$ .

Also note that

O and B, and between A and 
$$N = P(1-)$$
 hypotenuses of neht triengles

What is the geometric representation of multiplication by a number?

Example 7. Let A = (1, 2) and c = 3. Then

$$cA = (3, 6)$$

as in Fig. 7 (a).

Multiplication by 3 amounts to stretching A by 3. Similarly,  $\frac{1}{2}A$  amounts to stretching A by  $\frac{1}{2}$ , i.e. shrinking A to half its size. In general, if t is a number,

t > 0, we interpret tA as a point in the same direction as A from the origin, but t times the distance. In fact, we define A and B to have the same direction if there exists a number c > 0 such that A = cB. We emphasize that this means A and B have the same direction with respect to the origin. For simplicity of language, we omit the words "with respect to the origin."

Multiplication by a negative number reverses the direction. Thus -3Awould be represented as in Fig. 7 (b).

We define two vectors A, B (neither of which is zero) to have opposite directions if there is a number c < 0 such that cA = B. Thus when B = -A, then A, B have opposite direction.

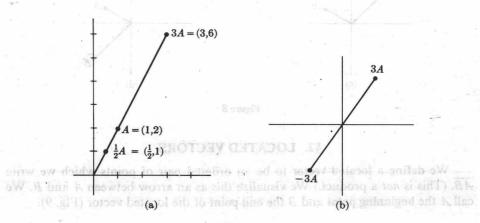


Figure 7

### **EXERCISES**

Find A + B, A - B, 3A, -2B in each of the following cases. Draw the points of Exercises 1 and 2 on a sheet of graph paper.

1. 
$$A = (2, -1), B = (-1, 1)$$
.

2. 
$$A = (-1, 3), B = (0, 4)$$

3. 
$$A = (2, -1, 5), B = (-1, 1, 1)$$

3. 
$$A = (2, -1, 5), B = (-1, 1, 1)$$
 4.  $A = (-1, -2, 3), B = (-1, 3, -4)$ 

5. 
$$A = (\pi, 3, -1), B = (2\pi, -3, 7)$$

5. 
$$A = (\pi, 3, -1), B = (2\pi, -3, 7)$$
 6.  $A = (15, -2, 4), B = (\pi, 3, -1)$ 

- 7. Let A = (1, 2) and B = (3, 1). Draw A + B, A + 2B, A + 3B, A B, A 2B, A - 3B on a sheet of graph paper.
- 8. Let A, B be as in Exercise 1. Draw the points A + 2B, A + 3B, A 2B, A 3B,  $A + \frac{1}{2}B$  on a sheet of graph paper.
- 9. Let A and B be as drawn in Fig. 8. Draw the point A B.