Index Theory for Hamiltonian Systems and Multiple Solution Problems

(哈密顿系统指标理论与多解问题)

Yujun Dong (董玉君)

SCIENCE PRESS
Beijing

Index Theory for Hamiltonian Systems and Multiple Solution Problems

(哈密顿系统指标理论与多解问题)

Yujun Dong (董玉君)

科学出版社
北京

Responsible Editors: Yanchao Zhao and Jingke Li

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-042542-3

To my parents and sisters

Preface

This book concerns index theory for linear Hamiltonian systems and multiple solutions for asymptotically linear Hamiltonian systems, as well as some related problems. There are two excellent books on Hamiltonian systems: *Convexity Methods in Hamiltonian Mechanics* by I. Ekeland and *Index Theory for Symplectic Paths with Applications* by Y. Long. Periodic solutions are the main topics in those books, whereas non-periodic solutions will be investigated in most parts of this book. Most contents are from my own research works during the past twenty years and I will try to write this book following the style of the famous book *Minimax Methods in Critical Point Theory with Applications to Differential Equations* by Paul H. Rabinowitz. An overview is given of the subject matter in Chapter 1 and a detailed study is carried out in the Chapters that follow.

I began to study the existence of solutions for Duffing equations(the simplest case of Hamiltonian systems) in 1988 following Prof. Qinde Zhou(who passed away in 1998 and left an excellent textbook[Zho]) at Jilin University and learnt index theory for symplectic paths with applications from Prof. Yiming Long during the period from 2000 to 2002 at Nankai University. Dr.s Yucheng Bu and Yingying Chen have read some parts of the book and found some errors. Editors Yanchao Zhao and Jingke Li have done a lot for publishing this book. NSFC(No.11171157) and PAPD (A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions) support my writing and publishing the book.

Yujun Dong
Nanjing
Aug. 2014

Contents

Chapter 1

An Overview

A focus of this book is the existence of solutions of Hamiltonian systems satisfying some boundary value conditions. It is well-known that any ODE (ordinary differential equation) satisfying an initial value condition has at least one solution provided the related function is continuous, e.g. if $G \subset \mathbf{R}^n$ is open and $f : (0,1) \times G \to \mathbf{R}^n$ is continuous, then the initial value problem

$$x' = f(t,x),$$
$$x(t_0) = x_0$$

has at least one solution for any $t_0 \in (0,1), x_0 \in G$. However boundary value problems are quite different. As an example consider

$$x'' + \lambda x = e(t), \tag{1.0.1}$$

$$x(0) = 0 = x(1). \tag{1.0.2}$$

(1.0.1)–(1.0.2) has no solutions if $\lambda = k^2\pi^2, e = \sin k\pi t, k \in \mathbf{N}^* \equiv \{1,2,3,\cdots\}$. But the problem has a unique solution if $\lambda \neq k^2\pi^2$ and $e \in C[0,1]$. A general version of (1.0.1) is the equation

$$x'' + h(t,x)x = e(t,x), \tag{1.0.3}$$

where $h, e : [0,1] \times \mathbf{R} \to \mathbf{R}$ are continuous, $h(t,x)$ is bounded and

(h_1) $k^2\pi^2 + \delta \leqslant h(t,x) \leqslant (k+1)^2\pi^2 - \delta, (t,x) \in [0,1] \times \mathbf{R}$.

A basic result is as follows.

Theorem 1.0.1 Suppose h satisfies (h_1) for some $k \in \mathbf{N}^*$ and $\delta > 0$. Then (1.0.2)–(1.0.3) has at least one solution.

This result is due to Lazer-Leach(LL) and is called the Lazer-Leach theorem. Its generalizations will be investigated in Chapters 2–8. In order to improve this result for (1.0.2)–(1.0.3) consider a classification problem for the equations

$$x'' + q(t)x = 0, \quad t \in (0,1), \tag{1.0.4}$$

where $q \in L^\infty[0,1]$.

Definition 1.0.1 We define $H_k = \{q \in L^\infty[0,1] | \text{the problem (1.0.4), (1.0.2)}$ has one nontrivial solution with exactly k zeros on $(0,1)\}$ for $k \in \mathbf{N} \equiv \{0,1,2,\cdots\}$.

Note that $k^2\pi^2 \in H_{k-1}$ for $k \in \mathbf{N}^*$ and $\mathbf{R}\backslash\{k^2\pi^2\}_{k=1}^\infty = (-\infty,\pi^2)\bigcup(\pi^2,4\pi^2)\bigcup(4\pi^2,9\pi^2)\bigcup\cdots$. Similarly $L^\infty[0,1]\backslash\bigcup\limits_{k=0}^{\infty} H_k = \bigcup\limits_{k=0}^{\infty} F_k$, where F_k is path-connected and $(\infty,\pi^2) \subset F_0$, $(k^2\pi^2,(k+1)^2\pi^2) \subset F_k$, $k \in \mathbf{N}^*$. This will be made precise in Section 2.2 by the so-called Prufer transformation and the Prufer equation as well as related results.

With this definition in hand the Lazer-Leach theorem has a generalized version.

Theorem 1.0.2 Suppose that there exist $q_j \in H_j, j = k, k+1$ for some $k \in \mathbf{N}$ and $\varepsilon > 0$ such that

(h_2) $q_k(t) + \varepsilon \leqslant h(t,x) \leqslant q_{k+1}(t) - \varepsilon$, $x \in \mathbf{R}$, a.e. $t \in [0,1]$.

Then (1.0.2)–(1.0.3) has at least one solution.

Definition 1.0.1 and Theorem 1.0.2 are from Dong[Do1]. These results and the definition of $F_k, k \in \mathbf{N}$ will be generalized to asymptotically positive linear Duffing equations in Chapter 3, to one-dimensional p-Laplacian equations in Chapter 4, to second order Hamiltonian systems in Chapter 5, to first order Hamiltonian systems in Chapter 6, and to operator equations in Chapters 7 and 8. Chapters 3 and 4 concern classification theories for positive linear systems or homogenous systems where the Prufer transformation will be used. However Chapters 5–8 concern index theories(i.e. classification theories) for linear systems where the Prufer transformation canot be used. To be more precise, consider the problem

$$x'' + B(t)x = 0, \tag{1.0.5}$$

$$x(0) = 0 = x(1), \tag{1.0.6}$$

where $B \in L^\infty([0,1],\mathcal{L}_s(\mathbf{R}^n))$ and as in Long[Lo4], $\mathcal{L}_s(\mathbf{R}^n)$ denotes the subspace of symmetric matrices in $\mathbf{R}^n$ with the induced norm in $\mathbf{R}^{n^2}$.

Definition 1.0.2 We define

$$\nu(B) = \text{the dimension of the solution subspace of (1.0.5)–(1.0.6)},$$

$$i(B) = \sum_{\lambda<0} \nu(B + \lambda I_n).$$

It is easy to check that $b \in H_k \bigcap \mathbf{R}$ if and only if $(\nu(b), i(b)) = (1,k)$ and $b \in F_k \bigcap \mathbf{R}$ if and only if $(\nu(b), i(b)) = (0,k)$. Part of this book concerns multiple solutions for Hamiltonian systems. $(i(B),\nu(B))$ was defined by Dong[Do4] motivated by Williom-Mawhin[WM], Ekeland[E] and Long[Lo2 and Lo3] which will be used to investigate multiple solutions for the problem

$$x'' + V'(t,x) = 0, \tag{1.0.7}$$
$$x(0) = 0 = x(1),$$

where $V \in C^1([0,1] \times \mathbf{R}^n)$ and $V'(t,x)$ denotes the gradient of $V(t,x)$ with respect to x.

Theorem 1.0.3 Assume that (i) $V \in C^2([0,1] \times \mathbf{R}^n)$ and there exist $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ satisfying $i(B_1) = i(B_2)$, $\nu(B_2) = 0$ and

$$B_1(t) \leqslant V''(t,x) \leqslant B_2(t), \quad \forall (t,x) \in [0,1] \times \mathbf{R}^{2n} \text{ with } |x| \geqslant r > 0,$$

(ii) $V'(t,0) \equiv 0$, $\nu(B_0) = 0$, where $B_0(t) \equiv V''(t,0)$ and $|i(B_1) - i(B_0)| \geqslant n$.

Then (1.0.6)–(1.0.7) has two nontrivial solutions.

Theorem 1.0.4 Assume that (i) $V \in C^2([0,1] \times \mathbf{R}^n)$ and there exist $B_1, B_2 \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$ satisfying $i(B_1) = i(B_2)$, $\nu(B_2) = 0$ and

$$B_1(t) \leqslant V''(t,x) \leqslant B_2(t), \quad \forall (t,x) \in [0,1] \times \mathbf{R}^{2n} \text{ with } |x| \geqslant r > 0,$$

(ii) $V'(t,0) \equiv 0$, $\nu(B_0) = 0$ where $B_0(t) \equiv V''(t,0)$, and

(iii) $V(t,-x) = V(t,x)$ for all $(t,x) \in [0,1] \times \mathbf{R}^n$.

Then (1.0.6)–(1.0.7) has at least $|i(B_1) - i(B_0)|$ distinct pairs of nonzero solutions.

These theorems will be proved by Morse theory[Ch2] and Ljusternik-Schnirelman theory[Ch3] in Chapter 5.

The following first order linear Hamiltonian system is a generalized form of (1.0.5):

$$-J\dot{z} - B(t)z = 0, \quad t \in (0,1), \tag{1.0.8}$$
$$x(0) = 0 = x(1),$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in $\mathbf{R}^n$, $z = (x,y), x, y \in \mathbf{R}^n$ and $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$. Here for simplicity we denote $(x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$ by (x,y) for $x, y \in \mathbf{R}^n$ and x^{T} denotes the transpose of x. In fact, let $z = (x, -\dot{x})$; then (1.0.5) is equivalent to (1.0.8) with B replaced by $\mathrm{diag}\{B, I_n\}$. The index $(i(B), \nu(B))$ concerning (1.0.5)–(1.0.6) can be generalized to the system (1.0.6) and (1.0.8).

Definition 1.0.3 For any $\lambda \in \mathbf{R}$ we define

$$i^f(\lambda I_{2n}) = nE\left[\frac{\lambda}{\pi}\right],$$

where as in Ekeland[E], $E[\alpha]$ is the integer a such that $a < \alpha \leqslant a+1$; and for any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$ we define

$$\nu^f(B) = \text{the dimension of the solution subspace of the system (1.0.6) and (1.0.8)},$$
$$i^f(B) = \sum_{\lambda \in [0,1)} \nu^f(\lambda B + (1-\lambda)cI_{2n}) + i^f(cI_{2n})$$

for all $c \in \mathbf{R}$ such that $B > cI_{2n}$.

The sum

$$\sum_{\lambda\in[0,1)} \nu^f(\lambda B_2 + (1-\lambda)B_1) \equiv I^f(B_1, B_2)$$

is called a relative Morse index between B_1 and B_2 if $B_1 < B_2$. I learnt this concept from Fei[F] and Long-Zhu[LoZ2, ZL]. The above version is from Dong[Do5].

Similar to Theorems 1.0.3 and 1.0.4, the index $(i^f(B), \nu^f(B))$ can be used to investigate the problem

$$\begin{aligned} &-J\dot{z} - H'(t,z) = 0, \\ &x(0) = 0 = x(1), \end{aligned}$$

where $z = (x, y), x, y \in \mathbf{R}^n$, $H \in C([0,1] \times \mathbf{R}^{2n})$ and $H'(t,z)$ denotes the gradient of $H(t,z)$ with respect to z.

It is easy to check that

$$\begin{aligned} &i^f(cI_{2n}) = i(c^2 I_n), \quad \forall c > 0, \\ &i^f(cI_{2n}) - i^f(c'I_{2n}) = I(c'I_{2n}, cI_{2n}), \quad \forall c > c' \end{aligned}$$

and

$$\nu(B) = \nu^f(\mathrm{diag}\{B, I_n\})$$

for all $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^{2n}))$. We will show in Chapter 6 that $i^f(B)$ is well-defined and for any $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$,

$$i(B) = i^f(\mathrm{diag}\{B, I_n\}).$$

The last topic in this book concerns index theory for linear operator equations

$$Ax - Bx = 0$$

and multiple solutions for

$$Ax - \nabla\Phi(x) = 0,$$

where A is self-adjoint in X with $\sigma(A) = \sigma_d(A)$, X is a real Hilbert space, $B \in \mathcal{L}_s(X)$ the space of bounded self-adjoint operaptors on X and $\Phi : Z \equiv D(|A|^{\frac{1}{2}}) \to \mathbf{R}$ is continuous differentiable and $\nabla\Phi \in C(Z, X)$ such that

$$\Phi'(x)y = (\nabla\Phi(x), y)_X$$

for all $x, y \in Z$. Chapter 7 concerns the case $\sigma(A) = \sigma_d(A)$ is bounded from below, but Chapter 8 $\sigma(A) = \sigma_d(A)$ unbounded from both above and below.

For related material we refer to [Ch1–3, E, EG, ELS, Lo4 and 5, MW, R].

Chapter 2

Duffing Equations(I)

In this chapter we give a proof for the Lazer-Leach theorem and generalize it to various cases for Duffing equations. The key ingredient is a classification theory for (2.2.1)–(2.2.2). The main results are from Dong[Do1 and Do2].

2.1 Lazer-Leach's theorem

Consider the problem

$$x'' + h(t,x)x = e(t,x), \quad t \in (0,1), \tag{2.1.1}$$

$$x(0) = 0 = x(1), \tag{2.1.2}$$

where $h, e : [0,1] \times \mathbf{R} \to \mathbf{R}$ are continuous. The main results of this section are the following theorems.

Theorem 2.1.1 Assume that $e(t,x)$ is bounded and $h(t,x)$ satisfies

(h_1) there are positive integer k and small enough constant $\delta > 0$ such that

$$k^2\pi^2 + \delta \leqslant h(t,x) \leqslant (k+1)^2\pi^2 - \delta, \quad (t,x) \in [0,1] \times \mathbf{R}.$$

Then (2.1.1)–(2.1.2) has one solution.

Theorem 2.1.2 Assume $e(t,x)$ is bounded and $h(t,x)$ satisfies

(h_2) there is a constant $\delta > 0$ such that

$$h(t,x) \leqslant \pi^2 - \delta, \quad (t,x) \in [0,1] \times \mathbf{R}.$$

Then (2.1.1)–(2.1.2) has one solution.

We need the following lemmas.

Lemma 2.1.1(Schauder Fixed Point Theorem) Let Ω be a closed convex subset of a Banach space. Suppose $T : \Omega \to \Omega$ and T is continuous and $T(\Omega)$ relatively compact. Then there is a point $x \in \Omega$ such that $T(x) = x$.

Lemma 2.1.2([Ch1, Theorem 1.1.5], [Ch3, Theorem 1.1.1] and [E, Theorem II.3.4]) Assume $f \in C([0,1] \times \mathbf{R}^n, \mathbf{R})$ and there exists $M > 0$ such that

$$|f(t,x)| \leqslant M(1 + |x|), \quad \forall\ (t,x) \in [0,1] \times \mathbf{R}^n,$$

where $|\cdot|$ denotes the usual norm in $\mathbf{R}^n$. Then the map $x(\cdot) \mapsto f(\cdot, x(\cdot))$ is continuous from $L^2([0,1], \mathbf{R}^n)$ to $L^2[0,1] \equiv L^2([0,1], \mathbf{R})$.

Proof of Theorem 2.1.1 Let $X = L^2[0,1]$ with the usual norm $\|x\|_2 \equiv \left(\int_0^1 x(t)^2 \mathrm{d}t\right)^{\frac{1}{2}}$ and associated inner product $(x,y)_2 \equiv \int_0^1 x(t)y(t)\mathrm{d}t$, and define $K_1 : X \to X$ by $x = K_1 u$ is the unique solution of the equation

$$-x'' = u$$

satisfying (2.1.2). Then

$$(K_1 x)(t) = \int_0^1 G_1(t,s)x(s)\mathrm{d}s,$$

where $G_1(t,s) = t(1-s)$ as $0 \leqslant t \leqslant s \leqslant 1$, $G_1(t,s) = s(1-t)$ as $0 \leqslant s \leqslant t \leqslant 1$. It is easy to check that K_1 is self-adjoint and compact and 0 is not an eigenvalue of K_1. By spectral theory there exist a basis $\{e_j\}_{j=1}^{\infty}$ and nonzero sequence $\mu_j \to 0$ such that $(e_j, e_k)_2 = \delta_{j,k}$, $K_1 e_j = \mu_j e_j$. So $\mu_j = \dfrac{1}{k^2\pi^2}$ and we can choose $e_j = \sqrt{2}\sin j\pi t$. Thus, $\{\sin j\pi t\}_{j=1}^{\infty}$ is an associated orthogonal basis of X. Let $x = K_2 u$ be the unique solution of

$$-x'' - \left(k^2 + k + \frac{1}{2}\right)\pi^2 x = u$$

satisfying (2.1.2). A direct computation shows that for any $u \in L^2[0,1]$,

$$(K_2 u)(t) = \int_0^1 G_2(t,s)u(s)\mathrm{d}s,$$

where $G_2(t,s) = \dfrac{1}{\mu \sin \mu} \sin \mu s \sin \mu(1-t)$ for $0 \leqslant s \leqslant t \leqslant 1$ and $G_2(t,s) = \dfrac{1}{\mu \sin \mu} \sin \mu t \sin \mu(1-s)$ for $0 \leqslant t \leqslant s \leqslant 1$, and $\mu = \left(k^2 + k + \dfrac{1}{2}\right)^{\frac{1}{2}}\pi$. In particular

$$(K_2 e_j)(t) = \left(j^2 - k^2 - k - \frac{1}{2}\right)^{-1} \pi^{-2} e_j(t).$$

Then for any $u \in X$ with $u = \sum_{j=1}^{\infty} c_j \sin j\pi t$,

$$(K_2 u)(t) = \sum_{j=1}^{\infty} c_j \left(j^2 - k^2 - k - \frac{1}{2}\right)^{-1} \pi^{-2} \sin j\pi t.$$

So $K_2 : X \to X$ is compact and

$$\|K_2 u\|_2 \leqslant \left(k + \frac{1}{2}\right)^{-1} \pi^{-2} \|u\|_2.$$

Set

$$(Nu)(t) = h(t, u(t))x(t) - \left(k^2 + k + \frac{1}{2}\right)\pi^2 u(t) - e(t, u(t))$$

for any $u \in X$. Then $N : X \to X$ is continuous via Lemma 2.1.2, and

$$\|Nu\|_2 \leqslant \left(\left(k + \frac{1}{2}\right)\pi^2 - \delta\right)\|u\|_2 + M, \quad \forall u \in X,$$

where $M > 0$ is a constant satisfying that $|e(t,x)| \leqslant M$ for all $(t,x) \in [0,1] \times \mathbf{R}$. Choose $R > 0$ such that $M \leqslant \delta R$. Then $K_2 N : \bar{U}_R \to \bar{U}_R$ has a fixed point $\bar{x}$ in $\bar{U}_R$ by Lemma 2.1.1, where $U_R = \{x \in X | \|x\|_2 < R\}$ and $\bar{U}_R$ is its closure. And this $\bar{x}$ is a solution of (2.1.1)–(2.1.2). The proof is complete. ∎

Proof of Theorem 2.1.2 Let $x = x(t)$ be a solution of (2.1.1)–(2.1.2). Multiplying (2.1.1) by x and integrating over $[0,1]$ yield

$$\|\dot{x}\|_2^2 \leqslant \int_0^1 (h(t,x)x - e(t,x))x\mathrm{d}t \leqslant (\pi^2 - \delta)\|x\|_2^2 + M\|x\|_2. \tag{2.1.3}$$

We need the following lemma.

Lemma 2.1.3(Poincare's inequality) For any $x \in H_0^1[0,1]$,

$$\|\dot{x}\|_2 \geqslant \pi\|x\|_2,$$

where $H_0^1[0,1] \equiv \{x \in H^1[0,1] | x(0) = 0 = x(1)\}$ and as usual $H^1[0,1] \equiv \{x : [0,1] \to \mathbf{R} | x(t)$ is absolutely continuous on $t \in [0,1]$ and $\dot{x} \in L^2[0,1]\}$ with the inner product $(x,y)_{1,2} \equiv \int_0^1 (\dot{x}(t)\dot{y}(t) + x(t)y(t))\mathrm{d}t$ and norm $\|x\|_{1,2} \equiv \left(\int_0^1 (\dot{x}(t)^2 + x(t)^2)\mathrm{d}t\right)^{\frac{1}{2}}$.

This lemma and (2.1.3) show that

$$\|x\|_0 \leqslant \|\dot{x}\|_2 \leqslant \frac{\pi M}{\delta} \equiv R,$$

where $\|x\|_0 \equiv \max\{|x(t)| : t \in [0,1]\}$. Set $h_1(t,x) = h(t,x)$ as $(t,x) \in [0,1] \times [-R,R]$, $h_1(t,x) = h(t, \pm R)$ as $\pm x \geqslant R$. It suffices to prove (2.1.1)–(2.1.2) with h replaced by h_1 has one solution. Let M_1 be a constant satisfying $M_1 \geqslant \pi^2$ and $-M_1 \leqslant h(t,x)$ for any $(t,x) \in [0,1] \times [-R,R]$. And let $2\mu^2 = M_1 - \pi^2 + \delta$ and $\mu > 0$. Let $x = K_3 u$ be the unique solution of the problem

$$-\ddot{x} + \mu^2 x = u(t),$$
$$x(0) = 0 = x(1).$$

Then

$$(K_3 u)(t) = \int_0^1 G_3(s,t)u(s)\mathrm{d}s,$$

where $G_3(s,t) = \dfrac{1}{\mu\sinh\mu}\sinh\mu(1-t)\sinh(\mu s)$ as $0 \leqslant s \leqslant t \leqslant 1$ and $G_3(s,t) = \dfrac{1}{\mu\sinh\mu}\sinh\mu(1-s)\sinh(\mu t)$ as $0 \leqslant t \leqslant s \leqslant 1$. Because $(j^2\pi^2+\mu^2)K_3e_j = e_j$, for any $u = \sum\limits_{j=1}^{\infty} c_je_j$,

$$(K_3u)(t) = \sum_{j=1}^{\infty}\frac{c_j}{j^2\pi^2+\mu^2}e_j.$$

Thus $\|K_3\| \leqslant \dfrac{1}{\pi^2+\mu^2}$. Let $(Nu)(t) = (h_1(t,u(t))+\mu^2)u(t) - e(t,u(t))$. We have

$$\|K_3Nu\| \leqslant M_2 + \delta_1\|u\|,$$

where $M_2 = \dfrac{M}{\pi^2+\mu^2}, \delta_1 = \dfrac{M_1+\pi^2-\delta}{M_1+\pi^2+\delta}$. Let $(1-\delta_1)R_1 \geqslant M_2$. Then $K_3N : \bar{U}_{R_1} \to \bar{U}_{R_1}$. By Lemma 2.1.1, there exists $\bar{x}$ such that $\bar{x} - K_3N\bar{x} = 0$. ∎

In order to prove Lemma 2.1.3, we need the following lemma.

Lemma 2.1.4 Assume $x \in H_0^1[0,1], f \in L^2[0,1]$ such that

$$\int_0^1 \dot{x}\dot{y}\mathrm{d}t - \int_0^1 yf\mathrm{d}t = 0, \quad \forall y \in H_0^1[0,1].$$

Then

$$\ddot{x} + f(t) = 0, \quad \text{a.e. } t \in (0,1), \quad x(0) = 0 = x(1).$$

Proof Set $e(t) = \dot{x}(t) + \int_0^t f(\tau)\mathrm{d}\tau - C$ such that $\int_0^1 e(t)\mathrm{d}t = 0$ and let $y(t) = \int_0^t e(\tau)\mathrm{d}\tau$. Then $\int_0^1 |e(t)|^2\mathrm{d}t = 0$ and $e(t) = 0$ for a.e. $t \in [0,1]$. Hence, the result follows. ∎

Proof of Lemma 2.1.3 Set $Z = H_0^1[0,1]$ with the inner product $(x,y)_Z \equiv \int_0^1 \dot{x}(t)\dot{y}(t)\mathrm{d}t$ and define $K : Z \to Z$ by $(Kx,y)_Z = \int_0^1 xy\mathrm{d}t$ for any $x,y \in Z$. K is self-adjoint and compact and 0 is not its eigenvalue. By spectral theory there is a basis $\{x_j\}$ of Z and a nonzero sequence $\lambda_j \to 0$ such that $(x_j,x_k)_Z = \delta_{jk}, Kx_j = \lambda_jx_j$. So

$$\int_0^1 x_jx\mathrm{d}t = \lambda_j\int_0^1 \dot{x}_j\dot{x}\mathrm{d}t$$

for all $x \in H_0^1[0,1]$. And hence

$$\ddot{x}_j + \frac{1}{\lambda_j}x_j = 0, \quad x_j(0) = 0 = x_j(1)$$