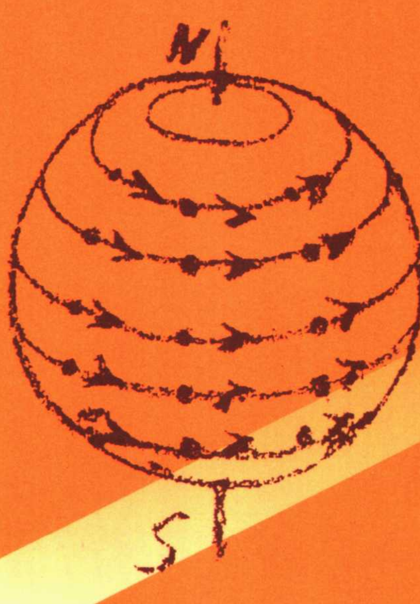
Arnol'd

Ordinary Differential Equations

常微分方程



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Vladimir I. Arnol'd

Ordinary Differential Equations

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Preface to the Third Edition

The first two chapters of this book have been thoroughly revised and significantly expanded. Sections have been added on elementary methods of integration (on homogeneous and inhomogeneous first-order linear equations and on homogeneous and quasi-homogeneous equations), on first-order linear and quasi-linear partial differential equations, on equations not solved for the derivative, and on Sturm's theorems on the zeros of second-order linear equations. Thus the new edition contains all the questions of the current syllabus in the theory of ordinary differential equations.

In discussing special devices for integration the author has tried throughout to lay bare the geometric essence of the methods being studied and to show how these methods work in applications, especially in mechanics. Thus to solve an inhomogeneous linear equation we introduce the delta-function and calculate the retarded Green's function; quasi-homogeneous equations lead to the theory of similarity and the law of universal gravitation, while the theorem on differentiability of the solution with respect to the initial conditions leads to the study of the relative motion of celestial bodies in neighboring orbits.

The author has permitted himself to include some historical digressions in this preface. Differential equations were invented by Newton (1642–1727). Newton considered this invention of his so important that he encoded it as an anagram whose meaning in modern terms can be freely translated as follows: "The laws of nature are expressed by differential equations."

One of Newton's fundamental analytic achievements was the expansion of all functions in power series (the meaning of a second, long anagram of Newton's to the effect that to solve any equation one should substitute the series into the equation and equate coefficients of like powers). Of particular importance here was the discovery of Newton's binomial formula (not with integer exponents, of course, for which the formula was known, for example, to Viète (1540–1603), but – what is particularly important – with fractional and negative exponents). Newton expanded all the elementary functions in "Taylor series" (rational functions, radicals, trigonometric, exponential, and logarithmic functions). This, together with a table of primitives compiled by Newton (which entered the modern textbooks of analysis almost unaltered), enabled him, in his words, to compare the areas of any figures "in half of a quarter of an hour."

Newton pointed out that the coefficients of his series were proportional to the successive derivatives of the function, but did not dwell on this, since he correctly considered that it was more convenient to carry out all the computations in analysis not by repeated differentiation, but by computing the first terms of a series. For Newton the connection between the coefficients of a series and the derivatives was more a means of computing derivatives than a means of constructing the series.

On of Newton's most important achievements is his theory of the solar system expounded in the *Mathematical Principles of Natural Philosophy* (the *Principia*) without using mathematical analysis. It is usually assumed that Newton discovered the law of universal gravitation using his analysis. In fact Newton deserves the credit only for proving that the orbits are ellipses (1680) in a gravitational field subject to the inverse-square law; the actual law of gravitation was shown to Newton by Hooke (1635–1703) (cf. § 8) and seems to have been guessed by several other scholars.

Modern physics begins with Newton's *Principia*. The completion of the formation of analysis as an independent scientific discipline is connected with the name of Leibniz (1646–1716). Another of Leibniz' grand achievements is the broad publicizing of analysis (his first publication is an article in 1684) and the development of its algorithms[1] to complete automatization: he thus discovered a method of teaching how to use analysis (and teaching analysis itself) to people who do not understand it at all – a development that has to be resisted even today.

Among the enormous number of eighteenth-century works on differential equations the works of Euler (1707–1783) and Lagrange (1736–1813) stand out. In these works the theory of small oscillations is first developed, and consequently also the theory of linear systems of differential equations; along the way the fundamental concepts of linear algebra arose (eigenvalues and eigenvectors in the n-dimensional case). The characteristic equation of a linear operator was long called the *secular equation*, since it was from just such an equation that the secular perturbations (long-term, i.e., slow in comparison with the annual motion) of planetary orbits were determined in accordance with Lagrange's theory of small oscillations. After Newton, Laplace and Lagrange and later Gauss (1777–1855) develop also the methods of perturbation theory.

When the unsolvability of algebraic equations in radicals was proved, Liouville (1809–1882) constructed an analogous theory for differential equations, establishing the impossibility of solving a variety of equations (including such classical ones as second-order linear equations) in elementary functions and quadratures. Later S. Lie (1842–1899), analyzing the problem of integration equations in quadratures, discovered the need for a detailed investigation of groups of diffeomorphisms (afterwards known as Lie groups) – thus from the

[1] Incidentally the concept of a matrix, the notation a_{ij}, the beginnings of the theory of determinants and systems of linear equations, and one of the first computing machines, are due to Leibniz.

theory of differential equations arose one of the most fruitful areas of modern mathematics, whose subsequent development was closely connected with completely different questions (Lie algebras had been studied even earlier by Poisson (1781–1840), and especially by Jacobi (1804–1851)).

A new epoch in the development of the theory of differential equations begins with the works of Poincaré (1854–1912), the "qualitative theory of differential equations," created by him, taken together with the theory of functions of a complex variable, lead to the foundation of modern topology. The qualitative theory of differential equations, or, as it is more frequently known nowadays, the theory of dynamical systems, is now the most actively developing area of the theory of differential equations, having the most important applications in physical science. Beginning with the classical works of A. M. Lyapunov (1857–1918) on the theory of stability of motion, Russian mathematicians have taken a large part in the development of this area (we mention the works of A. A. Andronov (1901–1952) on bifurcation theory, A. A. Andronov and L. S. Pontryagin on structural stability, N. M .Krylov (1879–1955) and N. N. Bogolyubov on the theory of averaging, A. N. Kolmogorov on the theory of perturbations of conditionally-periodic motions). A study of the modern achievements, of course, goes beyond the scope of the present book (one can become acquainted with some of them, for example, from the author's books, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York, 1983; *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978; and *Catastrophe Theory*, Springer-Verlag, New York, 1984).

The author is grateful to all the readers of earlier editions, who sent their comments, which the author has tried to take account of in revising the book, and also to D. V. Anosov, whose numerous comments promoted the improvement of the present edition.

V. I. Arnol'd

From the Preface to the First Edition

In selecting the material for this book the author attempted to limit the material to the bare minimum. The heart of the course is occupied by two circles of ideas: the theorem on the rectification of a vector field (equivalent to the usual existence, uniqueness, and differentiability theorems) and the theory of one-parameter groups of linear transformations (i.e., the theory of autonomous linear systems).

The applications of ordinary differential equations in mechanics are studied in more detail than usual. The equation of the pendulum makes its appearance at an early stage,; thereafter efficiency of the concepts introduced is always verified through this example. Thus the law of conservation of energy appears in the section on first integrals, the "small parameter method" is derived from the theorem on differentiation with respect to a parameter, and the theory of linear equations with periodic coefficients leads naturally to the study of the swing ("parametric resonance").

The exposition of many topics in the course differs widely from the traditional exposition. The author has striven throughout to make plain the geometric, qualitative side of the phenomena being studied. In accordance with this principle there are many figures in the book, but not a single complicated formula. On the other hand a whole series of fundamental concepts appears, concepts that remain in the shadows in the traditional coordinate presentation (the phase space and phase flows, smooth manifolds and bundles, vector fields and one-parameter diffeomorphism groups). The course could have been significantly shortened if these concepts had been assumed to be known. Unfortunately at present these topics are not included in courses of analysis or geometry. For that reason the author was obliged to expound them in some detail, assuming no previous knowledge on the part of the reader beyond the standard elementary courses of analysis and linear algebra.

The present book is based on a year-long course of lectures that the author gave to second-year mathematics majors at Moscow University during the years 1968–1970.

In preparing these lectures for publication the author received a great deal of help from R. I. Bogdanov. The author is grateful to him and all the students and colleagues who communicated their comments on the mimeographed text of the lectures (MGU, 1969). The author is also grateful to the reviewers D. V. Anosov and S. G. Krein for their attentive review of the manuscript.

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Frequently used notation

$\boldsymbol{R}$ – the set (group, field) of real numbers.
$\boldsymbol{C}$ – the set (group, field) of complex numbers.
$\boldsymbol{Z}$ – the set (group, ring) of integers.
$x \in X \subset Y$ – x is an element of the subset X of the set Y.
$X \cap Y$, $X \cup Y$ – the intersection and union of the sets X and Y.
$f : X \to Y$ – f is a mapping of the set X into the set Y.
$x \mapsto y$ – the mapping takes the point x to the point y.
$f \circ g$ – the composite of the mappings (g being applied first).
$\exists$, $\forall$ – there exists, for all.
* – a problem or theorem that is not obligatory (more difficult).
$\boldsymbol{R}^n$ – a vector space of dimension n over the field $\boldsymbol{R}$.

Other structures may be considered in ths set $\boldsymbol{R}^n$ (for example, an affine structure, a Euclidean structure, or the direct product of n lines). Usually this will be noted specifically ("the affine space $\boldsymbol{R}^n$", "the Euclidean space $\boldsymbol{R}^n$", "the coordinate space $\boldsymbol{R}^n$", and so forth).

Elements of a vector space are called *vectors*. Vectors are usually denoted by bold face letters ($\boldsymbol{v}$, $\boldsymbol{\xi}$, and so forth). Vectors of the coordinate space $\boldsymbol{R}^n$ are identified with n-tuples of numbers. We shall write, for example, $\boldsymbol{v} = (v_1, \ldots, v_n) = v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n$; the set of n vectors $\boldsymbol{e}_i$ is called a *coordinate basis* in $\boldsymbol{R}^n$.

We shall often encounter functions a real variable t called *time*. The derivative with respect to t is called *velocity* and is usually denoted by a dot over the letter: $\dot{x} = dx/dt$.

Contents

Chapter 1. Basic Concepts

§ 1. Phase Spaces

The theory of ordinary differential equations is one of the basic tools of mathematical science. This theory makes it possible to study all evolutionary processes that possess the properties of *determinacy, finite-dimensionality*, and *differentiability*. Before giving precise mathematical definitions, let us consider several examples.

1. Examples of Evolutionary Processes

A process is called *deterministic* if its entire future course and its entire past are uniquely determined by its state at the present time. The set of all states of the process is called the *phase space*.

Thus, for example, classical mechanics considers the motion of systems whose future and past are uniquely determined by the initial positions and initial velocities of all points of the system. The phase space of a mechanical system is the set whose elements are the sets of positions and velocities of all points of the given system.

The motion of particles in quantum mechanics is not described by a deterministic process. The propagation of heat is a semideterministic process: the future is determined by the present, but the past is not.

A process is called *finite-dimensional* if its phase space is finite-dimensional, i.e., if the number of parameters needed to describe its states is finite. Thus, for example, the Newtonian mechanics of systems consisting of a finite number of material points or rigid bodies belongs to this class. The dimension of the phase space of a system of n material points is $6n$, and that of a system of n rigid bodies is $12n$. The motions of a fluid studied in fluid mechanics, the vibrating processes of a string and a membrane, the propagation of waves in optics and acoustics are examples of processes that cannot be described using a finite-dimensional phase space.

A process is called *differentiable* if its phase space has the structure of a differentiable manifold, and the change of state with time is described by differentiable functions.

Thus, for example, the coordinates and velocities of the points of a mechanical system vary differentiably with time.