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ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS

ADVANCED BOOK PROGRAM

ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS

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To Our Wives

JESSICA AND REBA

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and

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PREFACE

The triumphant vindication of bold theories—are these not the pride and justification of our life's work?

—Sherlock Holmes, *The Valley of Fear*
Sir Arthur Conan Doyle

The main purpose of our book is to present and explain mathematical methods for obtaining approximate analytical solutions to differential and difference equations that cannot be solved exactly. Our objective is to help young and also established scientists and engineers to build the skills necessary to analyze equations that they encounter in their work. Our presentation is aimed at developing the insights and techniques that are most useful for attacking new problems. We do not emphasize special methods and tricks which work only for the classical transcendental functions; we do not dwell on equations whose exact solutions are known.

The mathematical methods discussed in this book are known collectively as asymptotic and perturbative analysis. These are the most useful and powerful methods for finding approximate solutions to equations, but they are difficult to justify rigorously. Thus, we concentrate on the most fruitful aspect of applied analysis; namely, obtaining the answer. We stress care but not rigor.

To explain our approach, we compare our goals with those of a freshman calculus course. A beginning calculus course is considered successful if the students have learned how to solve problems using calculus. It is not necessary for a student to understand the subtleties of interchanging limits, point-set topology, or measure theory to solve maximum-minimum problems, to compute volumes and areas, or to study the dynamics of physical systems. Asymptotics is a newer calculus, an approximate calculus, and its mathematical subtleties are as difficult for an advanced student as the subtleties of calculus are for a freshman. This volume teaches the new kind of approximate calculus necessary to solve hard problems approximately. We believe that our book is the first comprehensive book at the advanced undergraduate or beginning graduate level that has this kind of problem-solving approach to applied mathematics.

The minimum prerequisites for a course based on this book are a facility with calculus and an elementary knowledge of differential equations. Also, for a few advanced topics, such as the method of steepest descents, an awareness of complex variables is essential. This book has been used by us at Washington University and at M.I.T. in courses taken by engineering, science, and mathematics students normally including juniors, seniors, and graduate students.

We recognize that the readership of this book will be extremely diverse. Therefore, we have organized the book so that it will be useful to beginning students as well as to experienced researchers. First, this book is completely self-contained. We have included a review of ordinary differential equations and ordinary difference equations in Part I for those readers whose background is weak. There is also an Appendix of useful formulas so that it will rarely be necessary to consult outside reference books on special functions.

Second, we indicate the difficulty of every section by the three letters E (easy), I (intermediate), and D (difficult). We also use the letter T to indicate that the material has a theoretical as opposed to an applied or calculational slant. We have rated the material this way to help readers and teachers to select the level of material that is appropriate for their needs. We have included a large selection of exercises and problems at the end of each chapter. The difficulty and slant of each problem is also indicated by the letters E, I, D, and T. A good undergraduate course on mathematical methods can be based entirely on the sections and problems labeled E.

One of the novelties of this book is that we illustrate the results of our asymptotic analysis graphically by presenting many computer plots and tables which compare exact and approximate answers. These plots and tables should give the reader a feeling of just how well approximate analytical methods work. It is our experience that these graphs are an effective teaching device that strengthens the reader's belief that approximation methods can be usefully applied to the problems that he or she need to solve.

In this volume we are concerned only with functions of one variable. We hope some day to write a sequel to this book on partial differential equations.

We thank our many colleagues, especially at M.I.T., for their interest, suggestions, and contributions to our book, and our many students for their thoughtful and constructive criticism. We are grateful to Earl Cohen, Moshe Dubiner, Robert Keener, Lawrence Kells, Anthony Patera, Charles Peterson, Mark Preissler, James Shearer, Ellen Szeto, and Scot Tetrick for their assistance in preparing graphs and tables. We are particularly indebted to Jessica Bender for her tireless editorial assistance and to Shelley Bailey, Judi Cataldo, Joan Hill, and Darde Khan for helping us prepare a final manuscript. We both thank the National Science Foundation and the Sloan Foundation and one of us, S. A. O., thanks the Fluid Dynamics Branch of the Office of Naval Research for the support we have received during the preparation of this book. We also acknowledge the support of the National Center for Atmospheric Research for allowing us the use of their computers.

Carl M. Bender
Steven A. Orszag

CONTENTS†

† Each section is labeled according to difficulty: **(E)** = easy, **(I)** = intermediate, **(D)** = difficult. A section labeled **(T)** indicates that the material has a theoretical rather than an applied emphasis.

PART ONE

FUNDAMENTALS

I am afraid that I rather give myself away when I explain.
Results without causes are much more impressive.

Sherlock Holmes, *The Stock-Broker's Clerk*
Sir Arthur Conan Doyle

Part I of this book is a synopsis of exact methods for solving ordinary differential equations (Chap. 1) and ordinary difference equations (Chap. 2). Since our primary emphasis in later parts is on the approximate solution of such equations, it is important to review those exact methods that are currently known.

Our specific purpose with regard to differential equations is to refresh but not to introduce those concepts that would be learned in a low-level undergraduate course. Although Chap. 1 is self-contained in the sense that it begins with the most elementary aspects of the subject, the language and pace are appropriate for someone who has already had some experience in solving elementary differential equations. Our approach highlights applications rather than theory; we state theorems without proving them and stress methods for obtaining analytical solutions to equations.

The beauty of differential equations lies in their richness and variety. There is always a large class of equations which exhibits a new behavior or illustrates some counterintuitive notion. Unfortunately, many students, rather than enjoying the abundance of the subject, are confounded and appalled by it. To those who view the subject as an endless collection of unrelated methods, rules, and tricks, we say that the collection is actually finite; apart from transform methods (see the References), it is entirely contained in Chap. 1. The reader who masters the material in Chap. 1 will be fully prepared for any problems he or she may encounter. And to those mathematicians who prefer to study the general properties of a forest without having to examine individual trees, we are pleased to say that as we progress toward the approximate study of differential equations in Parts II, III, and IV our approach becomes far more general; approximate methods apply to much wider classes of equations than exact methods.

Our presentation in Chap. 2 is more elementary because most students are unfamiliar with difference equations. Our treatment of the subject emphasizes the parallels with differential equations and again stresses analytic methods of solution.

CHAPTER

ONE

ORDINARY DIFFERENTIAL EQUATIONS

Like all other arts, the Science of Deduction and Analysis is one which can only be acquired by long and patient study, nor is life long enough to allow any mortal to attain the highest possible perfection in it. Before turning to those moral and mental aspects of the matter which present the greatest difficulties, let the inquirer begin by mastering more elementary problems.

Sherlock Holmes, *A Study in Scarlet*
Sir Arthur Conan Doyle

(E) 1.1 ORDINARY DIFFERENTIAL EQUATIONS

An nth-order differential equation has the form

$$y^{(n)}(x) = F[x, y(x), y'(x), \ldots, y^{(n-1)}(x)], \tag{1.1.1}$$

where $y^{(k)} = d^k y/dx^k$. Equation (1.1.1) is a *linear* differential equation if F is a linear function of y and its derivatives (the explicit x dependence of F is still arbitrary). If (1.1.1) is linear, then the *general* solution $y(x)$ depends on n independent parameters called constants of integration; all solutions of a linear differential equation may be obtained by proper choice of these constants. If (1.1.1) is a nonlinear differential equation, then it also has a general solution which contains n constants of integration. However, there sometimes exist special additional solutions of nonlinear differential equations that cannot be obtained from the general solution for any choice of the integration constants. We omit a rigorous discussion of these fundamental properties of differential equations but illustrate them in the next three examples.

Example 1 *Separable equations.* Separable equations are the easiest differential equations to solve. An equation is *separable* if it is first order and the x and y dependences of F in (1.1.1) factor. The most general separable equation is

$$y'(x) = a(x)b(y). \tag{1.1.2}$$

Direct integration gives the general solution

$$\int^y \frac{dt}{b(t)} = \int^x a(s)\, ds + c_1, \tag{1.1.3}$$

where c_1 is a constant of integration. [The notation $\int^x a(s)\, ds$ stands for the antiderivative of $a(x)$.]

Linear equations have a simpler and more restricted range of possible behaviors than nonlinear equations, but they are an important class because they occur very frequently in the mathematical description of physical phenomena. Formally, a linear differential equation may be written as

$$Ly(x) = f(x), \tag{1.1.4}$$

where L is a linear differential operator:

$$L = p_0(x) + p_1(x)\frac{d}{dx} + \cdots + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n}. \tag{1.1.5}$$

It is conventional, although not necessary, to choose the coefficient of the highest derivative to be 1. If $f(x) \equiv 0$, the differential equation (1.1.4) is *homogeneous*; otherwise it is *inhomogeneous*.

Example 2 *Solution of a linear equation.* The general solution of the homogeneous linear equation

$$y'' - \frac{1+x}{x}y' + \frac{1}{x}y = 0 \tag{1.1.6}$$

is $y(x) = c_1 e^x + c_2(1 + x)$, which shows the explicit dependence on the two constants of integration c_1 and c_2. Every solution of (1.1.6) has this form.

Nonlinear equations have a richer mathematical structure than linear equations and are generally more difficult to solve in closed form. Nevertheless, the solution of many difficult-looking nonlinear equations is quite routine.

Example 3 *Solutions of nonlinear equations.* Two nonlinear differential equations which can be routinely solved (see Secs. 1.6 and 1.7) are the Riccati equation

$$y' = \frac{A^2}{x^4} - y^2, \qquad A \text{ is a constant}, \tag{1.1.7}$$

whose general solution is

$$y(x) = \frac{1}{x} + \frac{A}{x^2}\frac{c_1 - e^{2A/x}}{c_1 + e^{2A/x}} \tag{1.1.8}$$

and the equidimensional equation

$$y'' = yy'/x \tag{1.1.9}$$

whose general solution is

$$y(x) = 2c_1 \tan(c_1 \ln x + c_2) - 1. \tag{1.1.10}$$

There is a special solution to (1.1.9), namely $y = c_3$, where c_3 is an arbitrary constant, which cannot be obtained from the general solution in (1.1.10) for any choice of c_1 and c_2. (See Prob. 1.2.)

The rest of this chapter gives a brief theoretical discussion of the existence and uniqueness of solutions to initial- and boundary-value problems and surveys the elementary techniques for obtaining closed-form solutions of differential equations like those in the above three examples.

Systems of First-Order Equations

The general nth-order differential equation (1.1.1) is equivalent to a system of n first-order equations. To show this, we introduce the n dependent variables $y_k(x) = d^k y/dx^k$ ($k = 0, 1, 2, \ldots, n-1$). These variables satisfy the system of n first-order equations

$$\frac{d}{dx} y_k(x) = y_{k+1}(x), \qquad k = 0, \ldots, n-2,$$

$$\frac{d}{dx} y_{n-1}(x) = F[x, y_0, y_1, y_2, \ldots, y_{n-1}(x)].$$

Conversely, it is usually true that a system of n first-order equations

$$\frac{d}{dx} z_k = f_k(x, z_1, z_2, \ldots, z_n), \qquad k = 1, 2, \ldots, n, \tag{1.1.11}$$

can be transformed to a single equation of nth order. To construct an equivalent nth-order equation for $z_1(x)$, we differentiate (1.1.11) with respect to x, using the chain rule and (1.1.11) for dz_k/dx. We obtain

$$\frac{d^2}{dx^2} z_1 = \frac{\partial f_1}{\partial x} + \sum_{k=1}^{n} \frac{\partial f_1}{\partial z_k} f_k \equiv f_1^{(1)}(x, z_1, \ldots, z_n).$$

Repeating this process $(n-1)$ times we obtain n equations of the form

$$\frac{d^j}{dx_j} z_1 = f_1^{(j)}(x, z_1, \ldots, z_n), \qquad j = 1, \ldots, n, \tag{1.1.12}$$

where $f_1^{(0)} = f_1$ and $f_1^{(j+1)} = \partial f_1^{(j)}/\partial x + \sum_{k=1}^{n} (\partial f_1^{(j)}/\partial z_k) f_k$. If these n equations can be solved simultaneously to eliminate $z_2, z_3, \ldots, z_n$ as functions of x, z_1, dz_1/dx, $d^2 z_1/dx^2, \ldots, d^{n-1} z_1/dx^{n-1}$, then the system (1.1.11) has been transformed to an nth-order equation for z_1. Can you construct an example in which the equations (1.1.12) cannot be solved for $z_2, \ldots, z_n$?

(E) 1.2 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

A solution $y(x)$ to a differential equation is not uniquely determined by the differential equation alone; the values of the n independent constants of integration must also be specified. These constants of integration may be specified in several quite disparate ways. In an *initial-value problem* one specifies y and its first $n-1$ derivatives, $y', \ldots, y^{(n-1)}$, at *one* point $x = x_0$:

$$\begin{aligned} y(x_0) &= a_0, \\ y'(x_0) &= a_1, \ldots, \\ y^{(n-1)}(x_0) &= a_{n-1}. \end{aligned} \tag{1.2.1}$$