

INTERNATIONAL SERIES IN PURE AND APPLIED MATHEMATICS



ADVANCED MATHEMATICAL METHODS
FOR SCIENTISTS AND ENGINEERS



ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS

Carl M. Bender

*Professor of Physics
Washington University*

Steven A. Orszag

*Professor of Applied Mathematics
Massachusetts Institute of Technology*

McGRAW-HILL BOOK COMPANY

New York St. Louis San Francisco Auckland Bogotá Düsseldorf
Johannesburg London Madrid Mexico Montreal New Delhi
Panama Paris São Paulo Singapore Sydney Tokyo Toronto

ADVANCED MATHEMATICAL METHODS FOR SCIENTISTS AND ENGINEERS

Copyright © 1978 by McGraw-Hill, Inc. All rights reserved.
Printed in the United States of America. No part of this publication
may be reproduced, stored in a retrieval system, or transmitted, in any
form or by any means, electronic, mechanical, photocopying, recording, or
otherwise, without the prior written permission of the publisher.

1234567890 FGRFGR 78321098

Library of Congress Cataloging in Publication Data

Bender, Carl M.

Advanced mathematical methods for scientists and engineers.

(International series in pure and applied mathematics)

Bibliography: p.

Includes index.

1. Differential equations—Numerical solutions.
2. Difference equations—Numerical solutions.
3. Engineering mathematics. 4. Science—Mathematics.

I. Orszag, S., joint author. II. Title.

QA371.B43 519.4 77-29168

ISBN 0-07-004452-X

This book was set in Monophoto Times Mathematics.

The editor was Rose Ciofalo and the production supervisor was Jeanne Selzam.

Fairfield Graphics was printer and binder.

Problems from the William Lowell Putnam Mathematical Competitions are reproduced here
with permission of The Mathematical Association of America.

To Our Wives

JESSICA AND REBA

and Sons

MICHAEL AND DANIEL

and

MICHAEL, PETER, AND JONATHAN

The triumphant vindication of bold theories—are these not the pride and justification of our life's work?

—Sherlock Holmes, *The Valley of Fear*
Sir Arthur Conan Doyle

The main purpose of our book is to present and explain mathematical methods for obtaining approximate analytical solutions to differential and difference equations that cannot be solved exactly. Our objective is to help young and also established scientists and engineers to build the skills necessary to analyze equations that they encounter in their work. Our presentation is aimed at developing the insights and techniques that are most useful for attacking new problems. We do not emphasize special methods and tricks which work only for the classical transcendental functions; we do not dwell on equations whose exact solutions are known.

The mathematical methods discussed in this book are known collectively as asymptotic and perturbative analysis. These are the most useful and powerful methods for finding approximate solutions to equations, but they are difficult to justify rigorously. Thus, we concentrate on the most fruitful aspect of applied analysis; namely, obtaining the answer. We stress care but not rigor.

To explain our approach, we compare our goals with those of a freshman calculus course. A beginning calculus course is considered successful if the students have learned how to solve problems using calculus. It is not necessary for a student to understand the subtleties of interchanging limits, point-set topology, or measure theory to solve maximum-minimum problems, to compute volumes and areas, or to study the dynamics of physical systems. Asymptotics is a newer calculus, an approximate calculus, and its mathematical subtleties are as difficult for an advanced student as the subtleties of calculus are for a freshman. This volume teaches the new kind of approximate calculus necessary to solve hard problems approximately. We believe that our book is the first comprehensive book at the advanced undergraduate or beginning graduate level that has this kind of problem-solving approach to applied mathematics.

The minimum prerequisites for a course based on this book are a facility with calculus and an elementary knowledge of differential equations. Also, for a few

advanced topics, such as the method of steepest descents, an awareness of complex variables is essential. This book has been used by us at Washington University and at M.I.T. in courses taken by engineering, science, and mathematics students normally including juniors, seniors, and graduate students.

We recognize that the readership of this book will be extremely diverse. Therefore, we have organized the book so that it will be useful to beginning students as well as to experienced researchers. First, this book is completely self-contained. We have included a review of ordinary differential equations and ordinary difference equations in Part I for those readers whose background is weak. There is also an Appendix of useful formulas so that it will rarely be necessary to consult outside reference books on special functions.

Second, we indicate the difficulty of every section by the three letters E (easy), I (intermediate), and D (difficult). We also use the letter T to indicate that the material has a theoretical as opposed to an applied or calculational slant. We have rated the material this way to help readers and teachers to select the level of material that is appropriate for their needs. We have included a large selection of exercises and problems at the end of each chapter. The difficulty and slant of each problem is also indicated by the letters E, I, D, and T. A good undergraduate course on mathematical methods can be based entirely on the sections and problems labeled E.

One of the novelties of this book is that we illustrate the results of our asymptotic analysis graphically by presenting many computer plots and tables which compare exact and approximate answers. These plots and tables should give the reader a feeling of just how well approximate analytical methods work. It is our experience that these graphs are an effective teaching device that strengthens the reader's belief that approximation methods can be usefully applied to the problems that he or she need to solve.

In this volume we are concerned only with functions of one variable. We hope some day to write a sequel to this book on partial differential equations.

We thank our many colleagues, especially at M.I.T., for their interest, suggestions, and contributions to our book, and our many students for their thoughtful and constructive criticism. We are grateful to Earl Cohen, Moshe Dubiner, Robert Keener, Lawrence Kells, Anthony Patera, Charles Peterson, Mark Preissler, James Shearer, Ellen Szeto, and Scot Tetrick for their assistance in preparing graphs and tables. We are particularly indebted to Jessica Bender for her tireless editorial assistance and to Shelley Bailey, Judi Cataldo, Joan Hill, and Darde Khan for helping us prepare a final manuscript. We both thank the National Science Foundation and the Sloan Foundation and one of us, S. A. O., thanks the Fluid Dynamics Branch of the Office of Naval Research for the support we have received during the preparation of this book. We also acknowledge the support of the National Center for Atmospheric Research for allowing us the use of their computers.

Carl M. Bender
Steven A. Orszag

Preface xiii

PART I
FUNDAMENTALS

	1	Ordinary Differential Equations	3
(E)	1.1	Ordinary Differential Equations (definitions; introductory examples)	3
(E)	1.2	Initial-Value and Boundary-Value Problems (definitions; comparison of local and global analysis; examples of initial-value problems)	5
(TE)	1.3	Theory of Homogeneous Linear Equations (linear dependence and independence; Wronskians; well-posed and ill-posed initial-value and boundary-value problems)	7
(E)	1.4	Solutions of Homogeneous Linear Equations (how to solve constant-coefficient, equidimensional, and exact equations; reduction of order)	11
(E)	1.5	Inhomogeneous Linear Equations (first-order equations; variation of parameters; Green's functions; delta function; reduction of order; method of undetermined coefficients)	14
(E)	1.6	First-Order Nonlinear Differential Equations (methods for solving Bernoulli, Riccati, and exact equations; factoring; integrating factors; substitutions)	20
(I)	1.7	Higher-Order Nonlinear Differential Equations (methods to reduce the order of autonomous, equidimensional, and scale-invariant equations)	24

† Each section is labeled according to difficulty: **(E)** = easy, **(I)** = intermediate, **(D)** = difficult. A section labeled **(T)** indicates that the material has a theoretical rather than an applied emphasis.

(E)	1.8 Eigenvalue Problems	27	(examples of eigenvalue problems on finite and infinite domains)
(TE)	1.9 Differential Equations in the Complex Plane	29	(comparison of real and complex differential equations)
	Problems for Chapter 1	30	
2	Difference Equations	36	
(E)	2.1 The Calculus of Differences	36	(definitions; parallels between derivatives and differences, integrals, and sums)
(E)	2.2 Elementary Difference Equations	37	(examples of simple linear and nonlinear difference equations; gamma function; general first-order linear homogeneous and inhomogeneous equations)
(I)	2.3 Homogeneous Linear Difference Equations	40	(constant-coefficient equations; linear dependence and independence; Wronskians; initial-value and boundary-value problems; reduction of order; Euler equations; generating functions; eigenvalue problems)
(I)	2.4 Inhomogeneous Linear Difference Equations	49	(variation of parameters; reduction of order; method of undetermined coefficients)
(E)	2.5 Nonlinear Difference Equations	53	(elementary examples)
	Problems for Chapter 2	53	

PART II

LOCAL ANALYSIS

3	Approximate Solution of Linear Differential Equations	61	
(E)	3.1 Classification of Singular Points of Homogeneous Linear Equations	62	(ordinary, regular singular, and irregular singular points; survey of the possible kinds of behaviors of solutions)
(E)	3.2 Local Behavior Near Ordinary Points of Homogeneous Linear Equations	66	(Taylor series solution of first- and second-order equations; Airy equation)
(I)	3.3 Local Series Expansions About Regular Singular Points of Homogeneous Linear Equations	68	(methods of Fuchs and Frobenius; modified Bessel equation)
(E)	3.4 Local Behavior at Irregular Singular Points of Homogeneous Linear Equations	76	(failure of Taylor and Frobenius series; asymptotic relations; controlling factor and leading behavior; method of dominant balance; asymptotic series expansion of solutions at irregular singular points)

- (E) 3.5 Irregular Singular Point at Infinity 88
(theory of asymptotic power series; optimal asymptotic approximation; behavior of modified Bessel, parabolic cylinder, and Airy functions for large positive x)
- (E) 3.6 Local Analysis of Inhomogeneous Linear Equations 103
(illustrative examples)
- (TI) 3.7 Asymptotic Relations 107
(asymptotic relations for oscillatory functions; Airy functions and Bessel functions; asymptotic relations in the complex plane; Stokes phenomenon; subdominance)
- (TD) 3.8 Asymptotic Series 118
(formal theory of asymptotic power series; Stieltjes series and integrals; optimal asymptotic approximations; error estimates; outline of a rigorous theory of the asymptotic behavior of solutions to differential equations)
- Problems for Chapter 3 136
- 4 Approximate Solution of Nonlinear Differential Equations 146**
- (E) 4.1 Spontaneous Singularities 146
(comparison of the behaviors of solutions to linear and nonlinear equations)
- (E) 4.2 Approximate Solutions of First-Order Nonlinear Differential Equations 148
(several examples analyzed in depth)
- (I) 4.3 Approximate Solutions to Higher-Order Nonlinear Differential Equations 152
(Thomas–Fermi equation; first Painlevé transcendent; other examples)
- (I) 4.4 Nonlinear Autonomous Systems 171
(phase-space interpretation; classification of critical points; one- and two-dimensional phase space)
- (I) 4.5 Higher-Order Nonlinear Autonomous Systems 185
(brief, nontechnical survey of properties of higher-order systems; periodic, almost periodic, and random behavior; Toda lattice, Lorenz model, and other systems)
- Problems for Chapter 4 196
- 5 Approximate Solution of Difference Equations 205**
- (E) 5.1 Introductory Comments 205
(comparison of the behavior of differential and difference equations)
- (I) 5.2 Ordinary and Regular Singular Points of Linear Difference Equations 206
(classification of $n = \infty$ as an ordinary, a regular singular, or an irregular singular point; Taylor and Frobenius series at ∞)
- (E) 5.3 Local Behavior Near an Irregular Singular Point at Infinity: Determination of Controlling Factors 214
(three general methods)

- (E) 5.4 Asymptotic Behavior of $n!$ as $n \rightarrow \infty$: The Stirling Series 218
[asymptotic behavior of the gamma function $\Gamma(x)$ as $x \rightarrow \infty$ obtained from the difference equations $\Gamma(x+1) = x\Gamma(x)$]
 - (I) 5.5 Local Behavior Near an Irregular Singular Point at Infinity: Full Asymptotic Series 227
(Bessel functions of large order; Legendre polynomials of large degree)
 - (E) 5.6 Local Behavior of Nonlinear Difference Equations 233
(Newton's method and other nonlinear difference equations; statistical analysis of an unstable difference equation)
- Problems for Chapter 5 240

6 Asymptotic Expansion of Integrals 247

- (E) 6.1 Introduction 247
(integral representations of solutions to difference and differential equations)
 - (E) 6.2 Elementary Examples 249
(incomplete gamma function; exponential integral; other examples)
 - (E) 6.3 Integration by Parts 252
(many examples including some where the method fails)
 - (E) 6.4 Laplace's Method and Watson's Lemma 261
(modified Bessel, parabolic cylinder, and gamma functions; many other illustrative examples)
 - (I) 6.5 Method of Stationary Phase 276
(leading behavior of integrals with rapidly oscillating integrands)
 - (I) 6.6 Method of Steepest Descents 280
(steepest ascent and descent paths in the complex plane; saddle points; Stokes phenomenon)
 - (I) 6.7 Asymptotic Evaluation of Sums 302
(approximation of sums by integrals; Laplace's method for sums; Euler-Maclaurin sum formula)
- Problems for Chapter 6 306

PART III

PERTURBATION METHODS

7 Perturbation Series 319

- (E) 7.1 Perturbation Theory 319
(elementary introduction; application to polynomial equations and initial-value problems for differential equations)
- (E) 7.2 Regular and Singular Perturbation Theory 324
(classification of perturbation problems as regular or singular; introductory examples of boundary-layer, WKB, and multiple-scale problems)
- (I) 7.3 Perturbation Methods for Linear Eigenvalue Problems 330
(Rayleigh-Schrödinger perturbation theory)



- (D) 7.4 Asymptotic Matching 335
(matched asymptotic expansions; applications to differential equations, eigenvalue problems and integrals)
- (TD) 7.5 Mathematical Structure of Perturbative Eigenvalue Problems 350
(singularity structure of eigenvalues as functions of complex perturbing parameter; level crossing)
- Problems for Chapter 7 361
- 8 Summation of Series 368**
- (E) 8.1 Improvement of Convergence 368
(Shanks transformation; Richardson extrapolation; Riemann zeta function)
- (E) 8.2 Summation of Divergent Series 379
(Euler, Borel, and generalized Borel summation)
- (I) 8.3 Padé Summation 383
(one- and two-point Padé summation; generalized Shanks transformation; many numerical examples)
- (I) 8.4 Continued Fractions and Padé Approximants 395
(efficient methods for obtaining and evaluating Padé approximants)
- (TD) 8.5 Convergence of Padé Approximants 400
(asymptotic analysis of the rate of convergence of Padé approximants)
- (TD) 8.6 Padé Sequences for Stieltjes Functions 405
(monotonicity; convergence theory; moment problem; Carleman's condition)
- Problems for Chapter 8 410

PART IV

GLOBAL ANALYSIS

- 9 Boundary Layer Theory 417**
- (E) 9.1 Introduction to Boundary-Layer Theory 419
(linear and nonlinear examples)
- (E) 9.2 Mathematical Structure of Boundary Layers: Inner, Outer, and Intermediate Limits 426
(formal boundary-layer theory)
- (E) 9.3 Higher-Order Boundary Layer Theory 431
(uniformly valid global approximants to a simple boundary-value problem)
- (I) 9.4 Distinguished Limits and Boundary Layers of Thickness $\neq \varepsilon$ 435
(three illustrative examples)
- (I) 9.5 Miscellaneous Examples of Linear Boundary-Layer Problems 446
(third- and fourth-order differential equations; nested boundary layers)
- (D) 9.6 Internal Boundary Layers 455
(four cases including some for which boundary-layer theory fails)

- (I) 9.7 Nonlinear Boundary-Layer Problems 463
(a problem of Carrier; limit cycle of the Rayleigh oscillator)

Problems for Chapter 9 479

10 WKB Theory 484

- (E) 10.1 The Exponential Approximation for Dissipative and Dispersive Phenomena 484
(formal WKB expansion; relation to boundary-layer theory)
- (E) 10.2 Conditions for Validity of the WKB Approximation 493
(geometrical and physical optics)
- (E) 10.3 Patched Asymptotic Approximations: WKB Solution of Inhomogeneous Linear Equations 497
(WKB approximations to Green's functions)
- (I) 10.4 Matched Asymptotic Approximations: Solution of the One-Turning-Point Problem 504
(connection formula; Langer's solution; normalization methods)
- (I) 10.5 Two-Turning-Point Problems: Eigenvalue Condition 519
(approximate eigenvalues of Schrödinger equations)
- (D) 10.6 Tunneling 524
(reflection and transmission of waves through potential barriers)
- (D) 10.7 Brief Discussion of Higher-Order WKB Approximations 534
(second-order solution of one-turning-point problems; quantization condition to all orders)

Problems for Chapter 10 539

11 Multiple-Scale Analysis 544

- (E) 11.1 Resonance and Secular Behavior 544
(nonuniform convergence of regular perturbation expansions)
- (E) 11.2 Multiple-Scale Analysis 549
(formal theory; Duffing equation)
- (I) 11.3 Examples of Multiple-Scale Analysis 551
(damped oscillator; approach to a limit cycle; recovery of WKB and boundary layer approximations)
- (I) 11.4 The Mathieu Equation and Stability 560
(Floquet theory; stability boundaries of the Mathieu equation)

Problems for Chapter 11 566

Appendix—Useful Formulas 569

References 577

Index 581

PART ONE

FUNDAMENTALS

I am afraid that I rather give myself away when I explain.
Results without causes are much more impressive.

Sherlock Holmes, *The Stock-Broker's Clerk*
Sir Arthur Conan Doyle

Part I of this book is a synopsis of exact methods for solving ordinary differential equations (Chap. 1) and ordinary difference equations (Chap. 2). Since our primary emphasis in later parts is on the approximate solution of such equations, it is important to review those exact methods that are currently known.

Our specific purpose with regard to differential equations is to refresh but not to introduce those concepts that would be learned in a low-level undergraduate course. Although Chap. 1 is self-contained in the sense that it begins with the most elementary aspects of the subject, the language and pace are appropriate for someone who has already had some experience in solving elementary differential equations. Our approach highlights applications rather than theory; we state theorems without proving them and stress methods for obtaining analytical solutions to equations.

The beauty of differential equations lies in their richness and variety. There is always a large class of equations which exhibits a new behavior or illustrates some counterintuitive notion. Unfortunately, many students, rather than enjoying the abundance of the subject, are confounded and appalled by it. To those who view the subject as an endless collection of unrelated methods, rules, and tricks, we say that the collection is actually finite; apart from transform methods (see the References), it is entirely contained in Chap. 1. The reader who masters the material in Chap. 1 will be fully prepared for any problems he or she may encounter. And to those mathematicians who prefer to study the general properties of a forest without having to examine individual trees, we are pleased to say that as we progress toward the approximate study of differential equations in Parts II, III, and IV our approach becomes far more general; approximate methods apply to much wider classes of equations than exact methods.

2 FUNDAMENTALS

Our presentation in Chap. 2 is more elementary because most students are unfamiliar with difference equations. Our treatment of the subject emphasizes the parallels with differential equations and again stresses analytic methods of solution.

ORDINARY DIFFERENTIAL EQUATIONS

Like all other arts, the Science of Deduction and Analysis is one which can only be acquired by long and patient study, nor is life long enough to allow any mortal to attain the highest possible perfection in it. Before turning to those moral and mental aspects of the matter which present the greatest difficulties, let the inquirer begin by mastering more elementary problems.

Sherlock Holmes, *A Study in Scarlet*
Sir Arthur Conan Doyle

(E) 1.1 ORDINARY DIFFERENTIAL EQUATIONS

An n th-order differential equation has the form

$$y^{(n)}(x) = F[x, y(x), y'(x), \dots, y^{(n-1)}(x)], \quad (1.1.1)$$

where $y^{(k)} = d^k y/dx^k$. Equation (1.1.1) is a *linear* differential equation if F is a linear function of y and its derivatives (the explicit x dependence of F is still arbitrary). If (1.1.1) is linear, then the *general* solution $y(x)$ depends on n independent parameters called constants of integration; all solutions of a linear differential equation may be obtained by proper choice of these constants. If (1.1.1) is a nonlinear differential equation, then it also has a general solution which contains n constants of integration. However, there sometimes exist special additional solutions of nonlinear differential equations that cannot be obtained from the general solution for any choice of the integration constants. We omit a rigorous discussion of these fundamental properties of differential equations but illustrate them in the next three examples.

Example 1 *Separable equations.* Separable equations are the easiest differential equations to solve. An equation is *separable* if it is first order and the x and y dependences of F in (1.1.1) factor. The most general separable equation is

$$y'(x) = a(x)b(y). \quad (1.1.2)$$

Direct integration gives the general solution

$$\int^y \frac{dt}{b(t)} = \int^x a(s) ds + c_1, \quad (1.1.3)$$

where c_1 is a constant of integration. [The notation $\int^x a(s) ds$ stands for the antiderivative of $a(x)$.]

Linear equations have a simpler and more restricted range of possible behaviors than nonlinear equations, but they are an important class because they occur very frequently in the mathematical description of physical phenomena. Formally, a linear differential equation may be written as

$$Ly(x) = f(x), \quad (1.1.4)$$

where L is a linear differential operator:

$$L = p_0(x) + p_1(x) \frac{d}{dx} + \cdots + p_{n-1}(x) \frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n}. \quad (1.1.5)$$

It is conventional, although not necessary, to choose the coefficient of the highest derivative to be 1. If $f(x) \equiv 0$, the differential equation (1.1.4) is *homogeneous*; otherwise it is *inhomogeneous*.

Example 2 *Solution of a linear equation.* The general solution of the homogeneous linear equation

$$y'' - \frac{1+x}{x} y' + \frac{1}{x} y = 0 \quad (1.1.6)$$

is $y(x) = c_1 e^x + c_2(1+x)$, which shows the explicit dependence on the two constants of integration c_1 and c_2 . Every solution of (1.1.6) has this form.

Nonlinear equations have a richer mathematical structure than linear equations and are generally more difficult to solve in closed form. Nevertheless, the solution of many difficult-looking nonlinear equations is quite routine.

Example 3 *Solutions of nonlinear equations.* Two nonlinear differential equations which can be routinely solved (see Secs. 1.6 and 1.7) are the Riccati equation

$$y' = \frac{A^2}{x^4} - y^2, \quad A \text{ is a constant}, \quad (1.1.7)$$

whose general solution is

$$y(x) = \frac{1}{x} + \frac{A}{x^2} \frac{c_1 - e^{2A/x}}{c_1 + e^{2A/x}} \quad (1.1.8)$$

and the equidimensional equation

$$y'' = yy'/x \quad (1.1.9)$$

whose general solution is

$$y(x) = 2c_1 \tan(c_1 \ln x + c_2) - 1. \quad (1.1.10)$$

There is a special solution to (1.1.9), namely $y = c_3$, where c_3 is an arbitrary constant, which cannot be obtained from the general solution in (1.1.10) for any choice of c_1 and c_2 . (See Prob. 1.2.)

The rest of this chapter gives a brief theoretical discussion of the existence and uniqueness of solutions to initial- and boundary-value problems and surveys the elementary techniques for obtaining closed-form solutions of differential equations like those in the above three examples.

Systems of First-Order Equations

The general n th-order differential equation (1.1.1) is equivalent to a system of n first-order equations. To show this, we introduce the n dependent variables $y_k(x) = d^k y/dx^k$ ($k = 0, 1, 2, \dots, n-1$). These variables satisfy the system of n first-order equations

$$\begin{aligned} \frac{d}{dx} y_k(x) &= y_{k+1}(x), & k &= 0, \dots, n-2, \\ \frac{d}{dx} y_{n-1}(x) &= F[x, y_0, y_1, y_2, \dots, y_{n-1}(x)]. \end{aligned}$$

Conversely, it is usually true that a system of n first-order equations

$$\frac{d}{dx} z_k = f_k(x, z_1, z_2, \dots, z_n), \quad k = 1, 2, \dots, n, \quad (1.1.11)$$

can be transformed to a single equation of n th order. To construct an equivalent n th-order equation for $z_1(x)$, we differentiate (1.1.11) with respect to x , using the chain rule and (1.1.11) for dz_k/dx . We obtain

$$\frac{d^2}{dx^2} z_1 = \frac{\partial f_1}{\partial x} + \sum_{k=1}^n \frac{\partial f_1}{\partial z_k} f_k \equiv f_1^{(1)}(x, z_1, \dots, z_n).$$

Repeating this process ($n-1$) times we obtain n equations of the form

$$\frac{d^j}{dx^j} z_1 = f_1^{(j)}(x, z_1, \dots, z_n), \quad j = 1, \dots, n, \quad (1.1.12)$$

where $f_1^{(0)} = f_1$ and $f_1^{(j+1)} = \partial f_1^{(j)}/\partial x + \sum_{k=1}^n (\partial f_1^{(j)}/\partial z_k) f_k$. If these n equations can be solved simultaneously to eliminate z_2, z_3, \dots, z_n as functions of $x, z_1, dz_1/dx, d^2 z_1/dx^2, \dots, d^{n-1} z_1/dx^{n-1}$, then the system (1.1.11) has been transformed to an n th-order equation for z_1 . Can you construct an example in which the equations (1.1.12) cannot be solved for z_2, \dots, z_n ?

(E) 1.2 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

A solution $y(x)$ to a differential equation is not uniquely determined by the differential equation alone; the values of the n independent constants of integration must also be specified. These constants of integration may be specified in several quite disparate ways. In an *initial-value problem* one specifies y and its first $n-1$ derivatives, $y', \dots, y^{(n-1)}$, at one point $x = x_0$:

$$\begin{aligned} y(x_0) &= a_0, \\ y'(x_0) &= a_1, \dots, \\ y^{(n-1)}(x_0) &= a_{n-1}. \end{aligned} \quad (1.2.1)$$