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Jean-Pierre Serre

Galois Cohomology

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Translated from the French by Patrick Ion

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Foreword

This volume is an English translation of "Cohomologie Galoisienne". The original edition (Springer LN5, 1964) was based on the notes, written with the help of Michel Raynaud, of a course I gave at the Collège de France in 1962–1963. In the present edition there are numerous additions and one suppression: Verdier's text on the duality of profinite groups. The most important addition is the photographic reproduction of R. Steinberg's "Regular elements of semisimple algebraic groups", Publ. Math. I.H.E.S., 1965. I am very grateful to him, and to I.H.E.S., for having authorized this reproduction.

Other additions include:

- A proof of the Golod-Shafarevich inequality (Chap. I, App. 2).
- The "résumé de cours" of my 1991–1992 lectures at the Collège de France on Galois cohomology of $k(T)$ (Chap. II, App.).
- The "résumé de cours" of my 1990–1991 lectures at the Collège de France on Galois cohomology of semisimple groups, and its relation with abelian cohomology, especially in dimension 3 (Chap. III, App. 2).

The bibliography has been extended, open questions have been updated (as far as possible) and several exercises have been added.

In order to facilitate references, the numbering of propositions, lemmas and theorems has been kept as in the original 1964 text.

Jean-Pierre Serre
Harvard, Fall 1996

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Chapter I

Cohomology of profinite groups

§1. Profinite groups

1.1 Definition

A topological group which is the projective limit of finite groups, each given the discrete topology, is called a *profinite group*. Such a group is compact and totally disconnected.

Conversely:

Proposition 0. *A compact totally disconnected topological group is profinite.*

Let G be such a group. Since G is totally disconnected and locally compact, the open subgroups of G form a base of neighbourhoods of 1, cf. e.g. Bourbaki TG III, §4, n°6. Such a subgroup U has finite index in G since G is compact; hence its conjugates gUg^{-1} $(g \in G)$ are finite in number and their intersection V is both normal and open in G. Such V's are thus a base of neighbourhoods of 1; the map $G \to \varprojlim G/V$ is injective, continuous, and its image is dense; a compactness argument then shows that it is an isomorphism. Hence G is profinite.

The profinite groups form a category (the morphisms being continuous homomorphisms) in which infinite products and projective limits exist.

Examples.

1) Let L/K be a Galois extension of commutative fields. The Galois group $\mathrm{Gal}(L/K)$ of this extension is, by construction, the projective limit of the Galois groups $\mathrm{Gal}(L_i/K)$ of the finite Galois extensions L_i/K which are contained in L/K; thus it is a profinite group.

2) A compact analytic group over the p-adic field $\mathbf{Q}_p$ is profinite, when viewed as a topological group. In particular, $\mathbf{SL}_n(\mathbf{Z}_p), \mathbf{Sp}_{2n}(\mathbf{Z}_p), \ldots$ are profinite groups.

3) Let G be a discrete topological group, and let $\hat{G}$ be the projective limit of the finite quotients of G. The group $\hat{G}$ is called the profinite group *associated to* G; it is the separated completion of G for the topology defined by the subgroups of G which are of finite index; the kernel of $G \to \hat{G}$ is the intersection of all subgroups of finite index in G.

4) If M is a torsion abelian group, its dual $M^* = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$, given the topology of pointwise convergence, is a commutative profinite group. Thus one obtains the anti-equivalence (Pontryagin duality):

$$\text{torsion abelian groups} \iff \text{commutative profinite groups}$$

Exercises.

1) Show that a torsion-free commutative profinite group is isomorphic to a product (in general, an infinite one) of the groups $\mathbf{Z}_p$. [Use Pontryagin duality to reduce this to the theorem which says that every divisible abelian group is a direct sum of groups isomorphic to $\mathbf{Q}$ or to some $\mathbf{Q}_p/\mathbf{Z}_p$, cf. Bourbaki A VII.53, Exerc. 3.]

2) Let $G = \mathbf{SL}_n(\mathbf{Z})$, and let f be the canonical homomorphism

$$\hat{G} \longrightarrow \prod_p \mathbf{SL}_n(\mathbf{Z}_p).$$

(a) Show that f is surjective.

(b) Show the equivalence of the following two properties:

(b_1) f is an isomorphism;

(b_2) Each subgroup of finite index in $\mathbf{SL}_n(\mathbf{Z})$ is a congruence subgroup.

[These properties are known to be true for $n \neq 2$ and false for $n = 2$.]

1.2 Subgroups

Every closed subgroup H of a profinite group G is profinite. Moreover, the homogeneous space G/H is compact and totally disconnected.

Proposition 1. *If H and K are two closed subgroups of the profinite group G, with $H \supset K$, there exists a continuous section $s : G/H \to G/K$.*

(By "section" one means a map $s : G/H \to G/K$ whose composition with the projection $G/K \to G/H$ is the identity.)

We use two lemmas:

Lemma 1. *Let G be a compact group G, and let (S_i) be a decreasing filtration of G by closed subgroups. Let $S = \bigcap S_i$. The canonical map*

$$G/S \longrightarrow \varprojlim G/S_i$$

is a homeomorphism.

Indeed, this map is injective, and its image is dense; since the source space is compact, the lemma follows. (One could also invoke Bourbaki, TG III.59, cor. 3 to prop. 1.)

Lemma 2. *Proposition 1 holds if H/K is finite. If, moreover, H and K are normal in G, the extension*

$$1 \longrightarrow H/K \longrightarrow G/K \longrightarrow G/H \longrightarrow 1$$

splits (cf. §3.4) *over an open subgroup of G/H.*

Let U be an open normal subgroup of G such that $U \cap H \subset K$. The restriction of the projection $G/K \to G/H$ to the image of U is injective (and is a homomorphism whenever H and K are normal). Its inverse map is therefore a section over the image of U (which is open); one extends it to a section over the whole of G/H by translation.

Let us now prove prop. 1. One may assume $K = 1$. Let X be the set of pairs (S, s), where S is a closed subgroup of H and s is a continuous section $G/H \to G/S$. One gives X an ordering by saying that $(S, s) \geq (S', s')$ if $S \subset S'$ and if s' is the composition of s and $G/S \to G/S'$. If (S_i, s_i) is a totally ordered family of elements of X, and if $S = \bigcap S_i$, one has $G/S = \varprojlim G/S_i$ by Lemma 1; the s_i thus define a continuous section $s : G/H \to G/S$; one has $(S, s) \in X$. This shows that X is an inductively ordered set. By Zorn's Lemma, X contains a maximal element (S, s). Let us show that $S = 1$, which will complete the proof. If S were distinct from 1, then there would exist an open subgroup U of G such that $S \cap U \neq S$. Applying Lemma 2 to the triplet $(G, S, S \cap U)$, one would get a continuous section $G/S \to G/(S \cap U)$, and composing this with $s : G/H \to G/S$, would give a continuous section $G/H \to G/(S \cap U)$, in contradiction to the fact that (S, s) is maximal.

Exercises.

1) Let G be a profinite group acting continuously on a totally disconnected compact space X. Assume that G acts freely, i.e., that the stabilizer of each element of X is equal to 1. Show that there is a continuous section $X/G \to X$. [same proof as for prop. 1.]

2) Let H be a closed subgroup of a profinite group G. Show that there exists a closed subgroup G' of G such that $G = H \cdot G'$, which is minimal for this property.

1.3 Indices

A *supernatural number* is a formal product $\prod p^{n_p}$, where p runs over the set of prime numbers, and where n_p is an integer ≥ 0 or $+\infty$. One defines the product in the obvious way, and also the gcd and lcm of any family of supernatural numbers.

Let G be a profinite group, and let H be a closed subgroup of G. The *index* $(G : H)$ of H in G is defined as the lcm of the indices $(G/U : H/(H \cap U))$, where U runs over the set of open normal subgroups of G. It is also the lcm of the indices $(G : V)$ for open V containing H.

Proposition 2. (i) *If $K \subset H \subset G$ are profinite groups, one has*

$$(G : K) = (G : H) \cdot (H : K) .$$

(ii) *If (H_i) is a decreasing filtration of closed subgroups of G, and if $H = \bigcap H_i$, one has* $(G : H) = \operatorname{lcm}(G : H_i)$.

(iii) *In order that H be open in G, it is necessary and sufficient that* $(G : H)$ *be a natural number* (i.e., an element of $\mathbf{N}$).