Handbook of Statistics

VOLUME 1

Analysis of Variance

Edited by

P. R. Krishnaiah

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Preface

The field of statistics is growing at a rapid pace and the rate of publication of the books and papers on applied and theoretical aspects of statistics has been increasing steadily. The last decade has also witnessed the emergence of several new statistics journals to keep pace with the increase in research activity in statistics. With the advance of computer technology and the easy accessibility to statistical packages, more and more scientists in many disciplines have been using statistical techniques in data analysis. Statistics seems to be playing the role of a common denominator among all the scientists besides having profound influence on such matters like public policy. So, there is a great need to have comprehensive self-contained reference books to disseminate information on various aspects of statistical methodology and applications. The series *Handbook of Statistics* is started in an attempt to fulfill this need. Each volume in the series is devoted to a particular topic in statistics. The material in these volumes is essentially expository in nature and the proofs of the results are, in general, omitted. This series is addressed to the entire community of statisticians and scientists in various disciplines who use statistical methodology in their work. At the same time, special emphasis will be made on applications-oriented techniques with the applied statisticians in mind as the primary audience. It is believed that every scientist interested in statistics will be benefitted by browsing through these volumes.

The first volume of the series is devoted to the area of analysis of variance (ANOVA). The field of the ANOVA was developed by R. A. Fisher and others and has emerged as a very important branch of statistics. An attempt has been made to cover most of the useful techniques in univariate and multivariate ANOVA in this volume. Certain other aspects of the ANOVA not covered in this volume due to limitation of space are planned to be included in subsequent volumes since various branches of statistics are interlinked.

It is quite fitting that this volume is dedicated to the memory of the late H. Scheffé who made numerous important contributions to the field of

ANOVA. Scheffé's book *The Analysis of Variance* has significant impact on the field and his test for multiple comparisons of means of normal populations has been widely used.

I wish to thank Professors S. Das Gupta, N. L. Johnson, C. G. Khatri, K. V. Mardia and N. H. Timm for serving as members of the editorial board of this volume. Thanks are also due to the contributors to this volume and North-Holland Publishing Company for their excellent cooperation. Professors R. D. Bock, K. C. Chanda, S. Geisser, R. Gnanadesikan, S. J. Haberman, J. C. Lee, G. S. Mudholkar, M. D. Perlman, J. N. K. Rao, P. S. S. Rao and C. R. Rao were kind enough to review various chapters in this volume. I wish to express my appreciation to my distinguished colleague, Professor C. R. Rao, for his encouragement and inspiration.

P. R. Krishnaiah

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Estimation of Variance Components

C. Radhakrishna Rao and Jürgen Kleffe*

1. Introduction

The usual mixed linear model discussed in the literature on variance components is

$$Y = X\beta + U_1\phi_1 + \cdots + U_p\phi_p + \varepsilon \tag{1.1}$$

where X, $U_1,\ldots,U_p$ are known matrices, β is a fixed unknown vector parameter and $\phi_1,\ldots,\phi_p$, ε are unobservable random variables (r.v.'s) such that

$$E(\varepsilon)=0, \qquad E(\phi_i)=0, \qquad E(\phi_i\phi_j')=0, \quad i\neq j, \qquad E(\varepsilon\phi_i')=0,$$
$$E(\varepsilon\varepsilon')=\sigma_0^2 I_n, \qquad E(\phi_i\phi_i')=\sigma_i^2 I_{n_i}. \tag{1.2}$$

The unknown parameters $\sigma_0^2,\sigma_1^2,\ldots,\sigma_p^2$ are called variance components.

Some of the early uses of such models are due to Yates and Zacopancy (1935) and Cochran (1939) in survey sampling, Yates (1940) and Rao (1947, 1956) in combining intra and interblock information in design of experiments, Fairfield Smith (1936), Henderson (1950), Panse (1946) and Rao (1953) in the construction of selection indices in genetics, and Brownlee (1953) in industrial applications. A systematic study of the estimation of variance components was undertaken by Henderson (1953) who proposed three methods of estimation.

The general approach in all these papers was to obtain $p+1$ quadratic functions of Y, say $Y'Q_iY$, $i=1,\ldots,p+1$, which are invariant for translation of Y by $X\alpha$ where α is arbitrary, and solve the equations

$$Y'Q_iY = E(Y'Q_iY) = a_{i0}\sigma_0^2 + a_{i1}\sigma_1^2 + \cdots + a_{ip}\sigma_p^2, \quad i=0,1,\ldots,p. \tag{1.3}$$

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The method of choosing the quadratic forms was intuitive in nature (see Henderson, 1953) and did not depend on any stated criteria of estimation. The entries in the ANOVA table giving the sums of squares due to different effects were considered as good choices of the quadratic forms in general. The ANOVA technique provides good estimators in what are called balanced designs (see Anderson, 1975; Anderson and Crump, 1967) but, as shown by Seely (1975) such estimators may be inefficient in more general linear models. For a general discussion of Henderson's methods and their advantages (computational simplicity) and limitations (lack of uniqueness, inapplicability and inefficiency in special cases) the reader is referred to papers by Searle (1968, 1971), Seely (1975), Olsen et al. (1976) and Harville (1977, p.335).

A completely different approach is the ML (maximum likelihood) method initiated by Hartley and Rao (1967). They considered the likelihood of the unknown parameters β, $\sigma_0^2,\ldots,\sigma_p^2$ based on observed Y and obtained the likelihood equations by computing the derivatives of likelihood with respect to the parameters. Patterson and Thompson (1975) considered the marginal likelihood based on the maximal invariant of Y, i.e., only on $B'Y$ where $B=X^{\perp}$ (matrix orthogonal to X) and obtained what are called marginal maximum likelihood (MML) equations. Harville (1977) has given a review of the ML and MML methods and the computational algorithms associated with them.

ML estimators, though consistent may be heavily biased in small samples so that some caution is needed when they are used as estimates of individual parameters for taking decisions or for using them in the place of true values to obtain an efficient estimate of β. The problem is not acute if the exact distribution of the ML estimators is known, since in that case appropriate adjustments can be made in the individual estimators before using them. The general large sample properties associated with ML estimators are misleading in the absence of studies on the orders of sample sizes for which these properties hold in particular cases. The bias in MML estimators may not be large even in small samples. As observed earlier, the MML estimator is, by construction, a function of $B'Y$ the maximal invariant of Y. It turns out that even the full ML estimator is a function of $B'Y$ although the likelihood is based on Y. There are important practical cases where reduction of Y to $B'Y$ results in non-identifiability of individual parameters, in which case neither the ML nor the MML is applicable. The details are given in Section 5.

Rao (1970, 1971a,b, 1972, 1973) proposed a general method called MINQE (minimum norm quadratic estimation) the scope of which has been extended to cover a variety of situations by Focke and Dewess (1972), Kleffe (1975, 1976, 1977a,b, 1978, 1979), J.N.K. Rao (1973), Fuller and J.N.K. Rao (1978), P.S.R.S. Rao and Chaubey (1978), P.S.R.S. Rao (1977), Pukelsheim (1977, 1978a), Sinha and Wieand (1977) and Rao (1979). The method is applicable to a general linear model

$$Y = X\beta + \varepsilon, \qquad E(\varepsilon\varepsilon') = \theta_1 V_1 + \cdots + \theta_p V_p \tag{1.4}$$

where no structure need be imposed on ε and no restrictions are placed on θ_i or V_i. (In the model (1.1), $\theta_i \geqslant 0$ and V_i are non-negative definite.)

In the MINQE theory, we define what is called a natural estimator of a linear function $f'\theta$ of θ in terms of the unobservable r.v. ε in (1.4), say $\varepsilon' N\varepsilon$. Then the estimator $Y'AY$ in terms of the observable r.v. Y is obtained by minimizing the norm of the difference between the quadratic forms $\varepsilon' N\varepsilon$ and $Y'AY = (X\beta + \varepsilon)'A(X\beta + \varepsilon)$. The universality of the MINQE method as described in Rao (1979) and in this article arises from the following observations:

(a) It offers a wide scope in the choice of the norm depending on the nature of the model and prior information available.

(b) One or more restrictions such as invariance, unbiasedness and non-negative definiteness can be placed on $Y'AY$ depending on the desired properties of the estimators.

(c) The method is applicable in situations where ML and MML fail.

(d) There is an automatic provision for incorporating available prior information on the unknown parameters β and θ.

(e) Further, ML and MML estimators can be exhibited as iterated versions of suitably chosen MINQE's.

(f) The MINQE equation provides a natural numerical algorithm for computing the ML or MML estimator.

(g) For a suitable choice of the norm, the MINQ estimators provide minimum variance estimators of θ when Y is normally distributed.

It has been mentioned by some reviewers of the MINQE theory that the computations needed for obtaining the MINQ estimators are somewhat heavy. It is true that the closed form expressions given for MINQE's contain inverses of large order matrices, but they can be computed in a simple way in special cases that arise in practice. The computations in such cases are of the same order of magnitude as obtaining sums of squares in the ANOVA table appropriate for the linear model. It is certainly not true that the computation of MLE or MMLE is simpler than that of MINQE. Both may have the same order of complexity in the general case.

Recently, simple numerical techniques for computing MINQE's have been developed by Ahrens (1978), Swallow and Searl (1978) and Ahrens et al. (1979) for the unbalanced random ANOVA model and by Kleffe (1980) for several unbalanced two way classification models. Similar results for