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Benjamin Steinberg

Representation Theory of Finite Groups

An Introductory Approach

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An Introductory Approach

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To Nicholas Minoru

Preface

A student's first encounter with abstract algebra is usually in the form of linear algebra. This is typically followed by a course in group theory. Therefore, there is no reason a priori that an undergraduate could not learn group representation theory before taking a more advanced algebra course that covers module theory. In fact, group representation theory arguably could serve as a strong motivation for studying module theory over non-commutative rings. Also group representation theory has applications in diverse areas such as physics, statistics, and engineering, to name a few. One cannot always expect students from these varied disciplines to have studied module theory at the level needed for the "modern" approach to representation theory via the theory of semisimple algebras. Nonetheless, it is difficult, if not impossible, to find an undergraduate text on representation theory assuming little beyond linear algebra and group theory in prerequisites, and also assuming only a modest level of mathematical maturity.

During the Winter term of 2008, I taught a fourth year undergraduate/first year graduate course at Carleton University, which also included some third year students (and even one exceptionally talented second year student). After a bit of searching, I failed to find a text that matched the background and level of mathematical sophistication of my students. Faced with this situation, I decided to provide instead my own course notes, which have since evolved into this text. The goal was to present a gentle and leisurely introduction to group representation theory, at a level that would be accessible to students who have not yet studied module theory and who are unfamiliar with the more sophisticated aspects of linear algebra, such as tensor products. For this reason, I chose to avoid completely the Wedderburn theory of semisimple algebras. Instead, I have opted for a Fourier analytic approach. This sort of approach is normally taken in books with a more analytic flavor; such books, however, invariably contain material on the representation theory of compact groups, something else that I would consider beyond the scope of an undergraduate text. So here I have done my best to blend the analytic and the algebraic viewpoints in order to keep things accessible. For example, Frobenius reciprocity is treated from a character point of view to avoid use of the tensor product.

The only background required for most of this book is a basic knowledge of linear algebra and group theory, as well as familiarity with the definition of a ring. In particular, we assume familiarity with the symmetric group and cycle notation. The proof of Burnside's theorem makes use of a small amount of Galois theory (up to the fundamental theorem) and so should be skipped if used in a course for which Galois theory is not a prerequisite. Many things are proved in more detail than one would normally expect in a textbook; this was done to make things easier on undergraduates trying to learn what is usually considered graduate level material.

The main topics covered in this book include: character theory; the group algebra and Fourier analysis; Burnside's pq-theorem and the dimension theorem; permutation representations; induced representations and Mackey's theorem; and the representation theory of the symmetric group. The book ends with a chapter on applications to probability theory via random walks on groups.

It should be possible to present this material in a one semester course. Chapters 2–5 should be read by everybody; it covers the basic character theory of finite groups. The first two sections of Chap. 6 are also recommended for all readers; the reader who is less comfortable with Galois theory can then skip the last section of this chapter and move on to Chap. 7 on permutation representations, which is needed for Chaps. 8–10. Chapter 10, on the representation theory of the symmetric group, can be read immediately after Chap. 7. The final chapter, Chap. 11, provides an introduction to random walks on finite groups. It is intended to serve as a non-trivial application of representation theory, rather than as part of the core material of the book, and should therefore be taken as optional for those interested in the purely algebraic aspects of the theory. Chapter 11 can be read directly after Chap. 5, as it relies principally on Fourier analysis on abelian groups.

Although this book is envisioned as a text for an advanced undergraduate or introductory graduate level course, it is also intended to be of use for physicists, statisticians, and mathematicians who may not be algebraists, but need group representation theory for their work.

While preparing this book I have relied on a number of classical references on representation theory, including [5–7,10,15,20,21]. For the representation theory of the symmetric group I have drawn from [7,12,13,16,17,19]; the approach is due to James [17]. Good references for applications of representation theory to computing eigenvalues of graphs and random walks are [3,6,7]. Chapter 11, in particular, owes much of its presentation to [7] and [3]. Discrete Fourier analysis and its applications can be found in [3,7,22].

Thanks are due to the following people for their input and suggestions: Ariane Masuda, Paul Mezo, and Martin Steinberg.

Ottawa Benjamin Steinberg

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Chapter 1
Introduction

The representation theory of finite groups is a subject going back to the late eighteen hundreds. The pioneers in the subject were G. Frobenius, I. Schur, and W. Burnside. Modern approaches tend to make heavy use of module theory and the Wedderburn theory of semisimple algebras. But the original approach, which nowadays can be thought of as via discrete Fourier analysis, is much more easily accessible and can be presented, for instance, in an undergraduate course. The aim of this text is to exposit the essential ingredients of the representation theory of finite groups over the complex numbers assuming only knowledge of linear algebra and undergraduate group theory, and perhaps a minimal familiarity with ring theory.

The original purpose of representation theory was to serve as a powerful tool for obtaining information about finite groups via the methods of linear algebra, e.g., eigenvalues, inner product spaces, and diagonalization. The first major triumph of representation theory was Burnside's pq-theorem. This theorem states that a non-abelian group of order $p^a q^b$ with p, q prime cannot be simple, or equivalently, that every finite group of order $p^a q^b$ with p, q prime is solvable. It was not until much later [2, 14] that purely group theoretic proofs were found. Representation theory went on to play an indispensable role in the classification of finite simple groups.

However, representation theory is much more than just a means to study the structure of finite groups. It is also a fundamental tool with applications to many areas of mathematics and statistics, both pure and applied. For instance, sound compression is very much based on the fast Fourier transform for finite abelian groups. Fourier analysis on finite groups also plays an important role in probability and statistics, especially in the study of random walks on groups, such as card shuffling and diffusion processes [3, 7], and in the analysis of data [7, 8]; random walks are considered in the last chapter of the book. Applications of representation theory to graph theory, and in particular to the construction of expander graphs, can be found in [6]. Some applications along these lines, especially toward the computation of eigenvalues of Cayley graphs, are given in this text.

B. Steinberg, *Representation Theory of Finite Groups: An Introductory Approach*, Universitext, DOI 10.1007/978-1-4614-0776-8_1,

Chapter 2
Review of Linear Algebra

This chapter reviews the linear algebra that we shall assume throughout the book. Proofs of standard results are mostly omitted. The reader can consult a linear algebra text such as [4] for details. In this book all vector spaces considered will be finite dimensional over the field $\mathbb{C}$ of complex numbers.

2.1 Basic Definitions and Notation

This section introduces some basic notions from linear algebra. We start with some notation, not all of which belongs to linear algebra. Let V and W be vector spaces.

- If X is a set of vectors, then $\mathbb{C}X = \operatorname{Span} X$.
- $M_{mn}(\mathbb{C}) = \{m \times n \text{ matrices with entries in } \mathbb{C}\}$.
- $M_n(\mathbb{C}) = M_{nn}(\mathbb{C})$.
- $\operatorname{Hom}(V, W) = \{A\colon V \longrightarrow W \mid A \text{ is a linear map}\}$.
- $\operatorname{End}(V) = \operatorname{Hom}(V, V)$ (the *endomorphism ring* of V).
- $GL(V) = \{A \in \operatorname{End}(V) \mid A \text{ is invertible}\}$ (known as the *general linear group of* V).
- $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid A \text{ is invertible}\}$.
- The identity matrix/linear transformation is denoted I, or I_n if we wish to emphasize the dimension n.
- $\mathbb{Z}$ is the ring of integers.
- $\mathbb{N}$ is the set of non-negative integers.
- $\mathbb{Q}$ is the field of rational numbers.
- $\mathbb{R}$ is the field of real numbers.
- $\mathbb{Z}/n\mathbb{Z} = \{[0], \ldots, [n-1]\}$ is the ring of integers modulo n.
- R^* denotes the group of units (i.e., invertible elements) of a ring R.
- S_n is the group of permutations of $\{1, \ldots, n\}$, i.e., the *symmetric group* on n letters.
- The identity permutation is denoted Id.

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Elements of $\mathbb{C}^n$ will be written as n-tuples or as column vectors, as is convenient. If $A \in M_{mn}(\mathbb{C})$, we sometimes write A_{ij} for the entry in row i and column j. We may also write $A = (a_{ij})$ to mean the matrix with a_{ij} in row i and column j. If k, ℓ, m, and n are natural numbers, then matrices in $M_{mk,\ell n}(\mathbb{C})$ can be viewed as $m \times n$ block matrices with blocks in $M_{k\ell}(\mathbb{C})$. If we view an $mk \times \ell n$ matrix A as a block matrix, then we write $[A]_{ij}$ for the $k \times \ell$ matrix in the i, j block, for $1 \le i \le m$ and $1 \le j \le n$.

Definition 2.1.1 (Coordinate vector). If V is a vector space with basis $B = \{b_1, \ldots, b_n\}$ and $v = c_1 b_1 + \cdots + c_n b_n$ is a vector in V, then the *coordinate vector* of v with respect to the basis B is the vector $[v]_B = (c_1, \ldots, c_n) \in \mathbb{C}^n$. The map $T\colon V \longrightarrow \mathbb{C}^n$ given by $Tv = [v]_B$ is a vector space isomorphism that we sometimes call *taking coordinates* with respect to B.

Suppose that $T\colon V \longrightarrow W$ is a linear transformation and B, B' are bases for V, W, respectively. Let $B = \{v_1, \ldots, v_n\}$ and $B' = \{w_1, \ldots, w_m\}$. Then the *matrix of* T with respect to the bases B, B' is the $m \times n$ matrix $[T]_{B,B'}$ whose jth column is $[Tv_j]_{B'}$. In other words, if

$$Tv_j = \sum_{i=1}^{m} a_{ij} w_i,$$

then $[T]_{B,B'} = (a_{ij})$. When $V = W$ and $B = B'$, then we write simply $[T]_B$ for $[T]_{B,B}$.

The *standard basis* for $\mathbb{C}^n$ is the set $\{e_1, \ldots, e_n\}$ where e_i is the vector with 1 in the ith coordinate and 0 in all other coordinates. So when $n = 3$, we have

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1).$$

Throughout we will abuse the distinction between $\mathrm{End}(\mathbb{C}^n)$ and $M_n(\mathbb{C})$ and the distinction between $GL(\mathbb{C}^n)$ and $GL_n(\mathbb{C})$ by identifying a linear transformation with its matrix with respect to the standard basis.

Suppose $\dim V = n$ and $\dim W = m$. Then by choosing bases for V and W and sending a linear transformation to its matrix with respect to these bases we see that:

$$\mathrm{End}(V) \cong M_n(\mathbb{C});$$
$$GL(V) \cong GL_n(\mathbb{C});$$
$$\mathrm{Hom}(V, W) \cong M_{mn}(\mathbb{C}).$$

Notice that $GL_1(\mathbb{C}) \cong \mathbb{C}^*$ and so we shall always work with the latter. We indicate W is a subspace of V by writing $W \le V$.

If $W_1, W_2 \le V$, then by definition

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}.$$

This is the smallest subspace of V containing W_1 and W_2. If, in addition, $W_1 \cap W_2 = \{0\}$, then $W_1 + W_2$ is called a *direct sum*, written $W_1 \oplus W_2$. As vector spaces, $W_1 \oplus W_2 \cong W_1 \times W_2$ via the map $W_1 \times W_2 \longrightarrow W_1 \oplus W_2$ given by $(w_1, w_2) \mapsto w_1 + w_2$. In fact, if V and W are any two vector spaces, one can form their *external direct sum* by setting $V \oplus W = V \times W$. Note that

$$\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2.$$

More precisely, if B_1 is a basis for W_1 and B_2 is a basis for W_2, then $B_1 \cup B_2$ is a basis for $W_1 \oplus W_2$.

2.2 Complex Inner Product Spaces

Recall that if $z = a + bi \in \mathbb{C}$, then its *complex conjugate* is $\overline{z} = a - bi$. In particular, $z\overline{z} = a^2 + b^2 = |z|^2$. An *inner product* on V is a map

$$\langle \cdot, \cdot \rangle \colon V \times V \longrightarrow \mathbb{C}$$

such that, for $v, w, v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{C}$:

- $\langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle$;
- $\langle w, v \rangle = \overline{\langle v, w \rangle}$;
- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

A vector space equipped with an inner product is called an *inner product space*. The *norm* $\|v\|$ of a vector v in an inner product space is defined by $\|v\| = \sqrt{\langle v, v \rangle}$.

Example 2.2.1. The *standard inner product* on $\mathbb{C}^n$ is given by

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = \sum_{i=1}^{n} a_i \overline{b_i}.$$

Two important properties of inner products are the *Cauchy–Schwarz inequality*

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

and the *triangle inequality*

$$\|v + w\| \leq \|v\| + \|w\|.$$

Recall that two vectors v, w in an inner product space V are said to be *orthogonal* if $\langle v, w \rangle = 0$. A subset of V is called *orthogonal* if its elements are pairwise orthogonal. If, in addition, the norm of each vector is 1, the set is termed *orthonormal*. An orthogonal set of non-zero vectors is linearly independent. In particular, any orthonormal set is linearly independent.