Graduate Texts in Mathematics

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Nonsmooth Analysis and Control Theory

非光滑分析和控制论

Springer

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The authors dedicate this book:
to Gail, Julia, and Danielle;
to Sofia, Simeon, and Irina;
to Judy, Adam, and Sach; and
to Mary and Anna.

Preface

Pardon me for writing such a long letter; I had not the time to write a short one.

—Lord Chesterfield

Nonsmooth analysis refers to differential analysis in the absence of differentiability. It can be regarded as a subfield of that vast subject known as nonlinear analysis. While nonsmooth analysis has classical roots (we claim to have traced its lineage back to Dini), it is only in the last decades that the subject has grown rapidly. To the point, in fact, that further development has sometimes appeared in danger of being stymied, due to the plethora of definitions and unclearly related theories.

One reason for the growth of the subject has been, without a doubt, the recognition that nondifferentiable phenomena are more widespread, and play a more important role, than had been thought. Philosophically at least, this is in keeping with the coming to the fore of several other types of irregular and nonlinear behavior: catastrophes, fractals, and chaos.

In recent years, nonsmooth analysis has come to play a role in functional analysis, optimization, optimal design, mechanics and plasticity, differential equations (as in the theory of viscosity solutions), control theory, and, increasingly, in analysis generally (critical point theory, inequalities, fixed point theory, variational methods ...). In the long run, we expect its methods and basic constructs to be viewed as a natural part of differential analysis.

We have found that it would be relatively easy to write a very long book on nonsmooth analysis and its applications; several times, we did. We have now managed not to do so, and in fact our principal claim for this work is that it presents the essentials of the subject clearly and succinctly, together with some of its applications and a generous supply of interesting exercises. We have also incorporated in the text a number of new results which clarify the relationships between the different schools of thought in the subject. We hope that this will help make nonsmooth analysis accessible to a wider audience. In this spirit, the book is written so as to be used by anyone who has taken a course in functional analysis.

We now proceed to discuss the contents. Chapter 0 is an Introduction in which we allow ourselves a certain amount of hand-waving. The intent is to give the reader an *avant-goût* of what is to come, and to indicate at an early stage why the subject is of interest.

There are many exercises in Chapters 1 to 4, and we recommend (to the active reader) that they be done. Our experience in teaching this material has had a great influence on the writing of this book, and indicates that comprehension is proportional to the exercises done. The end-of-chapter problems also offer scope for deeper understanding. We feel no guilt in calling upon the results of exercises later as needed.

Chapter 1, on proximal analysis, should be done carefully by every reader of this book. We have chosen to work here in a Hilbert space, although the greater generality of certain Banach spaces having smooth norms would be another suitable context. We believe the Hilbert space setting makes for a more accessible theory on first exposure, while being quite adequate for later applications.

Chapter 2 is devoted to the theory of generalized gradients, which constitutes the other main approach (other than proximal) to developing nonsmooth analysis. The natural habitat of this theory is Banach space, which is the choice made. The relationship between these two principal approaches is now well understood, and is clearly delineated here. As for the preceding chapter, the treatment is not encyclopedic, but covers the important ideas.

In Chapter 3 we develop certain special topics, the first of which is value function analysis for constrained optimization. This topic is previewed in Chapter 0, and §3.1 is helpful, though not essential, in understanding certain proofs in the latter part of Chapter 4. The next topic, mean value inequalities, offers a glimpse of more advanced calculus. It also serves as a basis for the solvability results of the next section, which features the Graves–Lyusternik Theorem and the Lipschitz Inverse Function Theorem. Section 3.4 is a brief look at a *third* route to nonsmooth calculus, one that bases itself upon directional subderivates. It is shown that the salient points of this theory can be derived from the earlier results. We also present here a self-contained proof of Rademacher's Theorem. In §3.5 we develop some machinery that is used in the following chapter, notably measurable selection. We take a quick look at variational functionals, but by-and-large, the calculus of variations has been omitted. The final section of the chapter examines in more detail some questions related to tangency.

Chapter 4, as its title implies, is a self-contained introduction to the theory of control of ordinary differential equations. This is a biased introduction, since one of its avowed goals is to demonstrate virtually all of the preceding theory in action. It makes no attempt to address issues of modeling or of implementation. Nonetheless, most of the central issues in control are studied, and we believe that any serious student of mathematical control theory will find it essential to have a grasp of the tools that are developed here via nonsmooth analysis: invariance, viability, trajectory monotonicity, viscosity solutions, discontinuous feedback, and Hamiltonian inclusions. We believe that the unified and geometrically motivated approach presented here for the first time has merits that will continue to make themselves felt in the subject.

We now make some suggestions for the reader who does not have the time to cover all of the material in this book. If control theory is of less interest, then Chapters 1 and 2, together with as much of Chapter 3 as time allows, constitutes a good introduction to nonsmooth analysis. At the other extreme is the reader who wishes to do Chapter 4 virtually in its entirety. In that case, a jump to Chapter 4 directly after Chapter 1 is feasible; only occasional references to material in Chapters 2 and 3 is made, up to §4.8, and in such a way that the reader can refer back without difficulty. The two final sections of Chapter 4 have a greater dependence on Chapter 2, but can still be covered if the reader will admit the proofs of the theorems.

A word on numbering. All items are numbered in sequence within a section; thus Exercise 7.2 precedes Theorem 7.3, which is followed by Corollary 7.4. For references between two chapters, an extra initial digit refers to the chapter number. Thus a result that would be referred to as Theorem 7.3 within Chapter 1 would be invoked as Theorem 1.7.3 from within Chapter 4. All equation numbers are simple, as in (3), and start again at (1) at the beginning of each section (thus their effect is only local). A reference to §3 is to the third section of the current chapter, while §2.3 refers to the third section of Chapter 2.

A summary of our notational conventions is given in §0.5, and a Symbol Glossary appears in the Notes and Comments at the end of the book.

We would like to express our gratitude to the personnel of the Centre de Recherches Mathématiques (CRM) of l'Université de Montréal, and in particular to Louise Letendre, for their invaluable help in producing this book.

Finally, we learned as the book was going to press, of the death of our friend and colleague Andrei Subbotin. We wish to express our sadness at his passing, and our appreciation of his many contributions to our subject.

Francis Clarke, Lyon
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Introduction

Experts are not supposed to read this book at all.

—**R.P. Boas,** ***A Primer of Real Functions***

We begin with a motivational essay that previews a few issues and several techniques that will arise later in this book.

1 Analysis Without Linearization

Among the issues that routinely arise in mathematical analysis are the following three:

- **to minimize a function $f(x)$;**
- **to solve an equation $F(x) = y$ for x as a function of y; and**
- **to derive the stability of an equilibrium point x^* of a differential equation $\dot{x} = \varphi(x)$.**

None of these issues imposes by its nature that the function involved (f, F, or φ) be smooth (differentiable); for example, we can reasonably aim to minimize a function which is merely continuous, if growth or compactness is postulated.

Nonetheless, the role of derivatives in questions such as these has been central, due to the classical technique of *linearization*. This term refers to

the construction of a linear local approximation of a function by means of its derivative at a point. Of course, this approach requires that the derivative exists. When applied to the three scenarios listed above, linearization gives rise to familiar and useful criteria:

- at a minimum x, we have $f'(x) = 0$ (Fermat's Rule);
- if the $n \times n$ Jacobian matrix $F'(x)$ is nonsingular, then $F(x) = y$ is locally invertible (the Inverse Function Theorem); and
- if the eigenvalues of $\varphi'(x^*)$ have negative real parts, the equilibrium is locally stable.

The main purpose of this book is to introduce and motivate a set of tools and methods that can be used to address these types of issues, as well as others in analysis, optimization, and control, when the underlying data are not (necessarily) smooth.

In order to illustrate in a simple setting how this might be accomplished, and in order to make contact with what could be viewed as the first theorem in what has become known as *nonsmooth analysis*, let us consider the following question: to characterize in differential, thus local terms, the global property that a given continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is decreasing (i.e., $x \leq y \implies f(y) \leq f(x)$).

If the function f admits a continuous derivative f', then the integration formula

$$f(y) = f(x) + \int_x^y f'(t)\,dt$$

leads to a sufficient condition for f to be decreasing: that $f'(t)$ be nonpositive for each t. It is easy to see that this is necessary as well, so a satisfying characterization via f' is obtained.

If we go beyond the class of continuously differentiable functions, the situation becomes much more complex. It is known, for example, that there exists a *strictly* decreasing continuous f for which we have $f'(t) = 0$ almost everywhere. For such a function, the derivative appears to fail us, insofar as characterizing decrease is concerned.

In 1878, Ulysse Dini introduced certain constructs, one of which is the following (lower, right) *derivate*:

$$Df(x) := \liminf_{t \downarrow 0} \frac{f(x+t) - f(x)}{t}.$$

Note that $Df(x)$ can equal $+\infty$ or $-\infty$. It turns out that Df will serve our purpose, as we now see.

1.1. Theorem. *The continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is decreasing iff*

$$Df(x) \le 0 \ \forall x \in \mathbb{R}.$$

Although this result is well known, and in any case greatly generalized in a later chapter, let us indicate a nonstandard proof of it now, in order to bring out two themes that are central to this book: optimization and nonsmooth calculus.

Note first that $Df(x) \le 0$ is an evident necessary condition for f to be decreasing, so it is the sufficiency of this property that we must prove.

Let x, y be any two numbers with $x < y$. We will prove that for any $\delta > 0$, we have

$$\min\{ f(t)\colon y \le t \le y + \delta \} \le f(x). \tag{1}$$

This implies $f(y) \le f(x)$, as required.

As a first step in the proof of (1), let g be a function defined on $(x-\delta, y+\delta)$ with the following properties:

(a) g is continuously differentiable, $g(t) \ge 0$, $g(t) = 0$ iff $t = y$;

(b) $g'(t) < 0$ for $t \in (x-\delta, y)$ and $g'(t) \ge 0$ for $t \in [y, y+\delta)$; and

(c) $g(t) \to \infty$ as $t \downarrow x - \delta$, and also as $t \uparrow y + \delta$.

It is easy enough to give an explicit formula for such a function; we will not do so.

Now consider the minimization over $(x - \delta, y + \delta)$ of the function $f + g$; by continuity and growth, the minimum is attained at a point z. A necessary condition for a local minimum of a function is that its Dini derivate be nonnegative there, as is easily seen. This gives

$$D(f+g)(z) \ge 0.$$

Because g is smooth, we have the following fact (in nonsmooth calculus!):

$$D(f+g)(z) = Df(z) + g'(z).$$

Since $Df(z) \le 0$ by assumption, we derive $g'(z) \ge 0$, which implies that z lies in the interval $[y, y+\delta)$. We can now estimate the left side of (1) as follows:

$$\begin{aligned} \min\{ f(t)\colon y \le t \le y + \delta \} &\le f(z) \\ &\le f(z) + g(z) \text{ (since } g \ge 0) \\ &\le f(x) + g(x) \text{ (since } z \text{ minimizes } f+g). \end{aligned}$$