欧几里得原理十三本书

第3卷

The Thirteen Books of Euclid´s Elements

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[古希腊] 欧几里得 著

世界图书出版公司
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THE THIRTEEN BOOKS
OF
EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

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VOLUME III

BOOKS X—XIII AND APPENDIX

图书在版编目（C I P）数据

欧几里得原理十三本书=The Thirteen Books of Euclid's Elements. 第3卷 ：英文 /（古希腊）欧几里得（Euclid）著. 一影印本. 一北京 ：世界图书出版公司北京公司，2015. 2
ISBN 978-7-5100-9410-1

Ⅰ. ①欧… Ⅱ. ①欧… Ⅲ. ①欧氏几何一英文 Ⅳ. ①O184

中国版本图书馆 CIP 数据核字(2015)第037270号

欧几里得原理十三本书 第3卷
The Thirteen Books of Euclid's Elements Ⅲ

作　　者：[古希腊]欧几里得(Euclid)
策　　划：杜玉娟
出版统筹：李艺琳
责任编辑：张明德
封面设计：张瑞羚
出版发行：世界图书出版公司北京公司　　http://www.wpcbj.com.cn
地　　址：北京市朝内大街137号（邮编100010 电话010-64077922）
销　　售：各地新华书店及外文书店
印　　刷：北京财经印刷厂
开　　本：170㎜×230㎜
印　　张：35
字　　数：190千
版　　次：2015年5月第1版　2015年5月第1次印刷
书　　号：978-7-5100-9410-1
定　　价：80.00（元）

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BOOK X

INTRODUCTORY NOTE

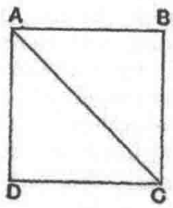
The discovery of the doctrine of incommensurables is attributed to Pythagoras. Thus Proclus says (*Comm. on Eucl.* I. p. 65, 19) that Pythagoras "discovered the theory of irrationals[1]"; and, again, the scholium on the beginning of Book X., also attributed to Proclus, states that the Pythagoreans were the first to address themselves to the investigation of commensurability, having discovered it by means of their observation of numbers. They discovered, the scholium continues, that not all magnitudes have a common measure. "They called all magnitudes measurable by the same measure commensurable, but those which are not subject to the same measure incommensurable, and again such of these as are measured by some other common measure commensurable with one another, and such as are not, incommensurable with the others. And thus by *assuming* their measures they referred everything to *different* commensurabilities, but, though they were different, even so (they proved that) not all magnitudes are commensurable with any. (They showed that) all magnitudes can be rational (ῥητά) and all irrational (ἄλογα) in a relative sense (ὡς πρός τι); hence the commensurable and the incommensurable would be for them *natural* (kinds) (φύσει), while the rational and irrational would rest on *assumption* or *convention* (θέσει)." The scholium quotes further the legend according to which "the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck," conjecturing that the authors of this story "perhaps spoke allegorically, hinting that everything irrational and formless is properly concealed, and, if any soul should rashly invade this region of life and lay it open, it would be carried away into the sea of becoming and be overwhelmed by its unresting currents." There would be a reason also for keeping the discovery of irrationals secret for the time in the fact that it rendered unstable so much of the groundwork of geometry as the Pythagoreans had based upon the imperfect theory of proportions which applied only to numbers. We have already, after Tannery, referred to the probability that the discovery of incommensurability must have necessitated a great recasting of the whole fabric of elementary geometry, pending the discovery of the general theory of proportion applicable to incommensurable as well as to commensurable magnitudes.

It seems certain that it was with reference to the length of the diagonal of a square or the hypotenuse of an isosceles right-angled triangle that Pythagoras made his discovery. Plato (*Theaetetus*, 147 D) tells us that Theodorus of Cyrene wrote about square roots (δυνάμεις), proving that the square roots of

[1] I have already noted (Vol. I. p. 351) that G. Junge (*Wann haben die Griechen das Irrationale entdeckt?*) disputes this, maintaining that it was the Pythagoreans, but not Pythagoras, who made the discovery. Junge is obliged to alter the reading of the passage of Proclus, on what seems to be quite insufficient evidence; and in any case I doubt whether the point is worth so much labouring.

three square feet and five square feet are not commensurable with that of one square foot, and so on, selecting each such square root up to that of 17 square feet, at which for some reason he stopped. No mention is here made of $\sqrt{2}$, doubtless for the reason that its incommensurability had been proved before, i.e. by Pythagoras. We know that Pythagoras invented a formula for finding right-angled triangles in rational numbers, and in connexion with this it was inevitable that he should investigate the relations between sides and hypotenuse in other right-angled triangles. He would naturally give special attention to the isosceles right-angled triangle; he would try to measure the diagonal, he would arrive at successive approximations, in rational fractions, to the value of $\sqrt{2}$; he would find that successive efforts to obtain an exact expression for it failed. It was however an enormous step to conclude that such exact expression was *impossible*, and it was this step which Pythagoras (or the Pythagoreans) made. We now know that the formation of the *side-* and *diagonal-*numbers explained by Theon of Smyrna and others was Pythagorean, and also that the theorems of Eucl. II. 9, 10 were used by the Pythagoreans in direct connexion with this method of approximating to the value of $\sqrt{2}$. The very method by which Euclid proves these propositions is itself an indication of their connexion with the investigation of $\sqrt{2}$, since he uses a figure made up of two isosceles right-angled triangles.

The actual method by which the Pythagoreans proved the incommensurability of $\sqrt{2}$ with unity was no doubt that referred to by Aristotle (*Anal. prior.* I. 23, 41 a 26—7), a *reductio ad absurdum* by which it is proved that, if the diagonal is commensurable with the side, it will follow that the same number is both odd and even. The proof formerly appeared in the texts of Euclid as X. 117, but it is undoubtedly an interpolation, and August and Heiberg accordingly relegate it to an Appendix. It is in substance as follows.



Suppose AC, the diagonal of a square, to be commensurable with AB, its side. Let $\alpha : \beta$ be their ratio expressed in the smallest numbers.

Then $\alpha > \beta$ and therefore necessarily > 1.

Now $$AC^2 : AB^2 = \alpha^2 : \beta^2,$$

and, since $$AC^2 = 2AB^2, \quad \text{[Eucl. I. 47]}$$

$$\alpha^2 = 2\beta^2.$$

Therefore α^2 is even, and therefore α is even.

Since $\alpha : \beta$ is in its lowest terms, it follows that β must be *odd*.

Put $$\alpha = 2\gamma;$$

therefore $$4\gamma^2 = 2\beta^2,$$

or $$\beta^2 = 2\gamma^2,$$

so that β^2, and therefore β, must be *even*.

But β was also odd:

which is impossible.

This proof only enables us to prove the incommensurability of the diagonal of a square with its side, or of $\sqrt{2}$ with unity. In order to prove the incommensurability of the sides of squares, one of which has *three* times the area of another, an entirely different procedure is necessary; and we find in fact that, even a century after Pythagoras' time, it was still necessary to use *separate* proofs (as the passage of the *Theaetetus* shows that Theodorus did) to establish the incommensurability with unity of $\sqrt{3}$, $\sqrt{5}$, ... up to $\sqrt{17}$.

This fact indicates clearly that the general theorem in Eucl. x. 9 that *squares which have not to one another the ratio of a square number to a square number have their sides incommensurable in length* was not arrived at all at once, but was, in the manner of the time, developed out of the separate consideration of special cases (Hankel, p. 103).

The proposition x. 9 of Euclid is definitely ascribed by the scholiast to Theaetetus. Theaetetus was a pupil of Theodorus, and it would seem clear that the theorem was not known to Theodorus. Moreover the Platonic passage itself (*Theaet.* 147 D sqq.) represents the young Theaetetus as striving after a general conception of what we call a *surd*. "The idea occurred to me, seeing that *square roots* (δυνάμεις) appeared to be unlimited in multitude, to try to arrive at one collective term by which we could designate all these square roots.... I divided number in general into two classes. The number which can be expressed as equal multiplied by equal (ἴσον ἰσάκις) I likened to a square in form, and I called it square and equilateral.... The intermediate number, such as three, five, and any number which cannot be expressed as equal multiplied by equal, but is either less times more or more times less, so that it is always contained by a greater and less side, I likened to an oblong figure and called an oblong number.... Such straight lines then as square the equilateral and plane number I defined as length (μῆκος), and such as square the oblong *square roots* (δυνάμεις), as not being commensurable with the others in length but only in the plane areas to which their squares are equal."

There is further evidence of the contributions of Theaetetus to the theory of incommensurables in a commentary on Eucl. x. discovered, in an Arabic translation, by Woepcke (*Mémoires présentés à l'Académie des Sciences*, XIV., 1856, pp. 658—720). It is certain that this commentary is of Greek origin. Woepcke conjectures that it was by Vettius Valens, an astronomer, apparently of Antioch, and a contemporary of Claudius Ptolemy (2nd cent. A.D.). Heiberg, with greater probability, thinks that we have here a fragment of the commentary of Pappus (*Euklid-studien*, pp. 169—71), and this is rendered practically certain by Suter (*Die Mathematiker und Astronomen der Araber und ihre Werke*, pp. 49 and 211). This commentary states that the theory of irrational magnitudes "had its origin in the school of Pythagoras. It was considerably developed by Theaetetus the Athenian, who gave proof, in this part of mathematics, as in others, of ability which has been justly admired. He was one of the most happily endowed of men, and gave himself up, with a fine enthusiasm, to the investigation of the truths contained in these sciences, as Plato bears witness for him in the work which he called after his name. As for the exact distinctions of the above-named magnitudes and the rigorous demonstrations of the propositions to which this theory gives rise, I believe that they were chiefly established by this mathematician; and, later, the great Apollonius, whose genius touched the highest point of excellence in mathematics, added to these discoveries a number of remarkable theories after many efforts and much labour.

"For Theaetetus had distinguished square roots [*puissances* must be the δυνάμεις of the Platonic passage] commensurable in length from those which are incommensurable, and had divided the well-known species of irrational lines after the different means, assigning the *medial* to geometry, the *binomial* to arithmetic, and the *apotome* to harmony, as is stated by Eudemus the Peripatetic.

"As for Euclid, he set himself to give rigorous rules, which he established, relative to commensurability and incommensurability in general; he made precise the definitions and the distinctions between rational and irrational magnitudes, he set out a great number of orders of irrational magnitudes, and finally he clearly showed their whole extent."

The allusion in the last words must be apparently to x. 115, where it is proved that from the *medial* straight line an unlimited number of other irrationals can be derived all different from it and from one another.

The connexion between the *medial* straight line and the geometric mean is obvious, because it is in fact the mean proportional between two rational straight lines "commensurable in square only." Since $\frac{1}{2}(x+y)$ is the *arithmetic* mean between x, y, the reference to it of the binomial can be understood. The connexion between the apotome and the harmonic mean is explained by some propositions in the second book of the Arabic commentary. The harmonic mean between x, y is $\frac{2xy}{x+y}$, and propositions of which Woepcke quotes the enunciations prove that, if a rational or a medial area has for one of its sides a *binomial* straight line, the other side will be an *apotome* of corresponding order (these propositions are generalised from Eucl. x. 111—4); the fact is that $\frac{2xy}{x+y} = \frac{2xy}{x^2-y^2}\cdot(x-y)$.

One other predecessor of Euclid appears to have written on irrationals, though we know no more of the work than its title as handed down by Diogenes Laertius[1]. According to this tradition, Democritus wrote περὶ ἀλόγων γραμμῶν καὶ ναστῶν β', *two Books on irrational straight lines and solids* (apparently). Hultsch (*Neue Jahrbücher für Philologie und Pädagogik*, 1881, pp. 578—9) conjectures that the true reading may be περὶ ἀλόγων γραμμῶν κλαστῶν, "on irrational broken lines." Hultsch seems to have in mind *straight* lines divided into two parts one of which is rational and the other irrational ("Aus einer Art von Umkehr des Pythagoreischen Lehrsatzes über das rechtwinklige Dreieck gieng zunächst mit Leichtigkeit hervor, dass man eine Linie construiren könne, welche als irrational zu bezeichnen ist, aber durch Brechung sich darstellen lässt als die Summe einer rationalen und einer irrationalen Linie"). But I doubt the use of κλαστός in the sense of breaking one straight line into parts; it should properly mean a bent line, i.e. two straight lines forming an angle or *broken short off* at their point of meeting. It is also to be observed that ναστόν is quoted as a Democritean word (opposite to κενόν) in a fragment of Aristotle (202). I see therefore no reason for questioning the correctness of the title of Democritus' book as above quoted.

I will here quote a valuable remark of Zeuthen's relating to the classification of irrationals. He says (*Geschichte der Mathematik im Altertum und Mittelalter*, p. 56) "Since such roots of equations of the second degree as are incommensurable with the given magnitudes cannot be expressed by means of the latter and of numbers, it is conceivable that the Greeks, in exact investigations, introduced no approximate values but worked on with the magnitudes they had found, which were represented by straight lines obtained by the construction corresponding to the solution of the equation. That is exactly the same thing which happens when we do not evaluate roots but content ourselves with expressing them by radical signs and other algebraical symbols. But, inasmuch as one straight line looks like another, the Greeks did not get

[1] Diog. Laert. IX. 47, p. 239 (ed. Cobet).

the same clear view of what they denoted (i.e. by simple inspection) as our system of symbols assures to us. For this reason it was necessary to undertake a classification of the irrational magnitudes which had been arrived at by successive solution of equations of the second degree." To much the same effect Tannery wrote in 1882 (*De la solution géométrique des problèmes du second degré avant Euclide* in *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 2ᵉ Série, IV. pp. 395—416). Accordingly Book X. formed a repository of results to which could be referred problems which depended on the solution of certain types of equations, quadratic and biquadratic but reducible to quadratics.

Consider the quadratic equations

$$x^2 \pm 2\alpha x \,.\, \rho \pm \beta \,.\, \rho^2 = 0,$$

where ρ is a rational straight line, and α, β are coefficients. Our quadratic equations in algebra leave out the ρ; but I put it in, because it has always to be remembered that Euclid's x is a *straight line*, not an algebraical quantity, and is therefore to be found in terms of, or in relation to, a certain assumed *rational straight line*, and also because with Euclid ρ may be not only of the form a, where a represents a units of length, but also of the form $\sqrt{\frac{m}{n}} \,.\, a$, which represents a length "commensurable in square only" with the unit of length, or $\sqrt{A}$ where A represents a number (not square) of units of *area*. The use therefore of ρ in our equations makes it unnecessary to multiply different *cases* according to the relation of ρ to the unit of length, and has the further advantage that, e.g., the expression $\rho \pm \sqrt{k} \,.\, \rho$ is just as general as the expression $\sqrt{k} \,.\, \rho \pm \sqrt{\lambda} \,.\, \rho$, since ρ covers the form $\sqrt{k} \,.\, \rho$, both expressions covering a length either commensurable in length, or "commensurable in square only," with the unit of length.

Now the *positive* roots of the quadratic equations

$$x^2 \pm 2\alpha x \,.\, \rho \pm \beta \,.\, \rho^2 = 0$$

can only have the following forms

$$\left.\begin{array}{l} x_1 = \rho\,(\alpha + \sqrt{\alpha^2 - \beta}),\ x_1' = \rho\,(\alpha - \sqrt{\alpha^2 - \beta}) \\ x_2 = \rho\,(\sqrt{\alpha^2 + \beta} + \alpha),\ x_2' = \rho\,(\sqrt{\alpha^2 + \beta} - \alpha) \end{array}\right\}.$$

The negative roots do not come in, since x must be a *straight line*. The omission however to bring in negative roots constitutes no loss of generality, since the Greeks would write the equation leading to negative roots in another form so as to make them positive, i.e. they would change the sign of x in the equation.

Now the positive roots x_1, x_1', x_2, x_2' may be classified according to the character of the coefficents α, β and their relation to one another.

I. Suppose that α, β do not contain any surds, i.e. are either integers or of the form m/n, where m, n are integers.

Now in the expressions for x_1, x_1' it may be that

(1) β is of the form $\frac{m^2}{n^2}\alpha^2$.

Euclid expresses this by saying that the square on $\alpha\rho$ exceeds the square on $\rho\sqrt{\alpha^2 - \beta}$ by the square on a straight line commensurable in length with $\alpha\rho$.

In this case x_1 is, in Euclid's terminology, a *first binomial* straight line,

and x_1' a *first apotome*.

(2) In general, β not being of the form $\frac{m^2}{n^2}\alpha^2$,

x_1 is a *fourth binomial*,

x_1' a *fourth apotome*.

Next, in the expressions for x_2, x_2' it may be that

(1) β is equal to $\frac{m^2}{n^2}(\alpha^2+\beta)$, where m, n are integers, i.e. β is of the form

$$\frac{m^2}{n^2-m^2}\alpha^2.$$

Euclid expresses this by saying that the square on $\rho\sqrt{\alpha^2+\beta}$ exceeds the square on $\alpha\rho$ by the square on a straight line commensurable in length with $\rho\sqrt{\alpha^2+\beta}$.

In this case x_2 is, in Euclid's terminology, a *second binomial*,

x_2' a *second apotome*.

(2) In general, β not being of the form $\frac{m^2}{n^2-m^2}\alpha^2$,

x_2 is a *fifth binomial*,

x_2' a *fifth apotome*.

II. Now suppose that α is of the form $\sqrt{\frac{m}{n}}$, where m, n are integers, and let us denote it by $\sqrt{\lambda}$.

Then in this case

$$x_1=\rho(\sqrt{\lambda}+\sqrt{\lambda-\beta}),\ x_1'=\rho(\sqrt{\lambda}-\sqrt{\lambda-\beta}),$$
$$x_2=\rho(\sqrt{\lambda+\beta}+\sqrt{\lambda}),\ x_2'=\rho(\sqrt{\lambda+\beta}-\sqrt{\lambda}).$$

Thus x_1, x_1' are of the same form as x_2, x_2'.

If $\sqrt{\lambda-\beta}$ in x_1, x_1' is not surd but of the form m/n, and if $\sqrt{\lambda+\beta}$ in x_2, x_2' is not surd but of the form m/n, the roots are comprised among the forms already shown, the first, second, fourth and fifth binomials and apotomes.

If $\sqrt{\lambda-\beta}$ in x_1, x_1' is surd, then

(1) we may have β of the form $\frac{m^2}{n^2}\lambda$, and in this case

x_1 is a *third binomial* straight line,

x_1' a *third apotome*;

(2) in general, β not being of the form $\frac{m^2}{n^2}\lambda$,

x_1 is a *sixth binomial* straight line,

x_1' a *sixth apotome*.

With the expressions for x_2, x_2' the distinction between the third and sixth binomials and apotomes is of course the distinction between the cases

(1) in which $\beta=\frac{m^2}{n^2}(\lambda+\beta)$, or β is of the form $\frac{m^2}{n^2-m^2}\lambda$,

and (2) in which β is not of this form.

If we take the square root of the product of ρ and each of the six binomials and six apotomes just classified, i.e.

$$\rho^2(\alpha\pm\sqrt{\alpha^2-\beta}),\ \rho^2(\sqrt{\alpha^2+\beta}\pm\alpha),$$

in the six different forms that each may take, we find six new irrationals with a positive sign separating the two terms, and six corresponding irrationals with a negative sign. These are of course roots of the equations

$$x^4 \pm 2\alpha x^2 . \rho^2 \pm \beta . \rho^4 = 0.$$

These irrationals really come before the others in Euclid's order (x. 36—41 for the positive sign and x. 73—78 for the negative sign). As we shall see in due course, the straight lines actually found by Euclid are

1. $\rho \pm \sqrt{k} . \rho$, the *binomial* (ἡ ἐκ δύο ὀνομάτων)
and the *apotome* (ἀποτομή),

which are the positive roots of the biquadratic (reducible to a quadratic)

$$x^4 - 2(1+k)\rho^2 . x^2 + (1-k)^2 \rho^4 = 0.$$

2. $k^{\frac{1}{4}}\rho \pm k^{\frac{3}{4}}\rho$, the *first bimedial* (ἐκ δύο μέσων πρώτη)
and the *first apotome of a medial* (μέσης ἀποτομὴ πρώτη),

which are the positive roots of

$$x^4 - 2\sqrt{k}(1+k)\rho^2 . x^2 + k(1-k)^2 \rho^4 = 0.$$

3. $k^{\frac{1}{4}}\rho \pm \dfrac{\sqrt{\lambda}}{k^{\frac{1}{4}}}\rho$, the *second bimedial* (ἐκ δύο μέσων δευτέρα)
and the *second apotome of a medial* (μέσης ἀποτομὴ δευτέρα),

which are the positive roots of the equation

$$x^4 - 2\frac{k+\lambda}{\sqrt{k}}\rho^2 . x^2 + \frac{(k-\lambda)^2}{k}\rho^4 = 0.$$

4. $\dfrac{\rho}{\sqrt{2}}\sqrt{1 + \dfrac{k}{\sqrt{1+k^2}}} \pm \dfrac{\rho}{\sqrt{2}}\sqrt{1 - \dfrac{k}{\sqrt{1+k^2}}}$,
the *major* (irrational straight line) (μείζων)
and the *minor* (irrational straight line) (ἐλάσσων),

which are the positive roots of the equation

$$x^4 - 2\rho^2 . x^2 + \frac{k^2}{1+k^2}\rho^4 = 0.$$

5. $\dfrac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}+k} \pm \dfrac{\rho}{\sqrt{2(1+k^2)}}\sqrt{\sqrt{1+k^2}-k}$,
the "*side*" *of a rational plus a medial* (area) (ῥητὸν καὶ μέσον δυναμένη)
and the "*side*" *of a medial minus a rational area* (in the Greek ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα),

which are the positive roots of the equation

$$x^4 - \frac{2}{\sqrt{1+k^2}}\rho^2 . x^2 + \frac{k^2}{(1+k^2)^2}\rho^4 = 0,$$

6. $\dfrac{\lambda^{\frac{1}{4}}\rho}{\sqrt{2}}\sqrt{1 + \dfrac{k}{\sqrt{1+k^2}}} \pm \dfrac{\lambda^{\frac{1}{4}}\rho}{\sqrt{2}}\sqrt{1 - \dfrac{k}{\sqrt{1+k^2}}}$,
the "*side*" *of the sum of two medial areas* (ἡ δύο μέσα δυναμένη)
and the "*side*" *of a medial minus a medial area* (in the Greek ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα),

which are the positive roots of the equation

$$x^4 - 2\sqrt{\lambda} . x^2\rho^2 + \lambda\frac{k^2}{1+k^2}\rho^4 = 0.$$

The above facts and formulae admit of being stated in a great variety of ways according to the notation and the particular letters used. Consequently the summaries which have been given of Eucl. x. by various writers differ much in appearance while expressing the same thing in substance. The first summary in algebraical form (and a very elaborate one) seems to have been that of Cossali (*Origine, trasporto in Italia, primi progressi in essa dell' Algebra*, Vol. II. pp. 242—65) who takes credit accordingly (p. 265). In 1794 Meier Hirsch published at Berlin an *Algebraischer Commentar über das zehente Buch der Elemente des Euklides* which gives the *contents* in algebraical form but fails to give any indication of Euclid's methods, using modern forms of proof only. In 1834 Poselger wrote a paper, *Ueber das zehnte Buch der Elemente des Euklides*, in which he pointed out the defects of Hirsch's reproduction and gave a summary of his own, which however, though nearer to Euclid's form, is difficult to follow in consequence of an elaborate system of abbreviations, and is open to the objection that it is not algebraical enough to enable the character of Euclid's irrationals to be seen at a glance. Other summaries will be found (1) in Nesselmann, *Die Algebra der Griechen*, pp. 165—84; (2) in Loria, *Il periodo aureo della geometria greca*, Modena, 1895, pp. 40—9; (3) in Christensen's article "Ueber Gleichungen vierten Grades im zehnten Buch der Elemente Euklids" in the *Zeitschrift für Math. u. Physik (Historisch-literarische Abtheilung)*, XXXIV. (1889), pp. 201—17. The only summary in English that I know is that in the *Penny Cyclopaedia*, under "Irrational quantity," by De Morgan, who yielded to none in his admiration of Book x. "Euclid investigates," says De Morgan, "every possible variety of lines which can be represented by $\sqrt{(\sqrt{a} \pm \sqrt{b})}$, a and b representing two commensurable lines....This book has a completeness which none of the others (not even the fifth) can boast of: and we could almost suspect that Euclid, having arranged his materials in his own mind, and having completely elaborated the 10th Book, wrote the preceding books after it and did not live to revise them thoroughly."

Much attention was given to Book x. by the early algebraists. Thus Leonardo of Pisa (fl. about 1200 A.D.) wrote in the 14th section of his *Liber Abaci* on the theory of irrationalities (*de tractatu binomiorum et recisorum*), without however (except in treating of irrational trinomials and cubic irrationalities) adding much to the substance of Book x.; and, in investigating the equation

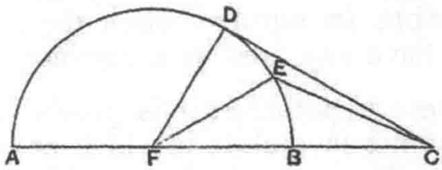
$$x^3 + 2x^2 + 10x = 20,$$

propounded by Johannes of Palermo, he proved that none of the irrationals in Eucl. x. would satisfy it (Hankel, pp. 344—6, Cantor, II_1, p. 43). Luca Paciuolo (about 1445—1514 A.D.) in his algebra based himself largely, as he himself expressly says, on Euclid x. (Cantor, II_1, p. 293). Michael Stifel (1486 or 1487 to 1567) wrote on irrational numbers in the second Book of his *Arithmetica integra*, which Book may be regarded, says Cantor (II_1, p. 402), as an elucidation of Eucl. x. The works of Cardano (1501—76) abound in speculations regarding the irrationals of Euclid, as may be seen by reference to Cossali (Vol. II., especially pp. 268—78 and 382—99); the character of the various odd and even powers of the binomials and apotomes is therein investigated, and Cardano considers in detail of what particular forms of equations, quadratic, cubic, and biquadratic, each class of Euclidean irrationals can be roots. Simon Stevin (1548—1620) wrote a *Traité des incommensurables grandeurs en laquelle est sommairement déclaré le contenu du Dixiesme Livre d'Euclide* (*Oeuvres mathématiques*, Leyde, 1634, pp. 219 sqq.); he speaks thus of the book: "La difficulté du dixiesme Livre d'Euclide est à plusieurs devenue en horreur, voire jusque à l'appeler la croix des mathématiciens, matière trop dure à digérer, et en la quelle n'aperçoivent aucune utilité," a passage quoted by Loria (*Il periodo aureo della geometria greca*, p. 41).

It will naturally be asked, what use did the Greek geometers actually make of the theory of irrationals developed at such length in Book x.? The answer is that Euclid himself, in Book XIII., makes considerable use of the second portion of Book x. dealing with the irrationals affected with a negative sign, the *apotomes* etc. One object of Book XIII. is to investigate the relation of the sides of a pentagon inscribed in a circle and of an icosahedron and dodecahedron inscribed in a sphere to the diameter of the circle or sphere respectively, supposed rational. The connexion with the regular pentagon of a straight line cut in extreme and mean ratio is well known, and Euclid first proves (XIII. 6) that, if a *rational* straight line is so divided, the parts are the irrationals called *apotomes*, the lesser part being a *first apotome*. Then, on the assumption that the diameters of a circle and sphere respectively are rational, he proves (XIII. 11) that the side of the inscribed regular pentagon is the irrational straight line called *minor*, as is also the side of the inscribed icosahedron (XIII. 16), while the side of the inscribed dodecahedron is the irrational called an *apotome* (XIII. 17).

Of course the investigation in Book x. would not have been complete if it had dealt only with the irrationals affected with a *negative* sign. Those affected with the positive sign, the *binomials* etc., had also to be discussed, and we find both portions of Book x., with its nomenclature, made use of by Pappus in two propositions, of which it may be of interest to give the enunciations here.

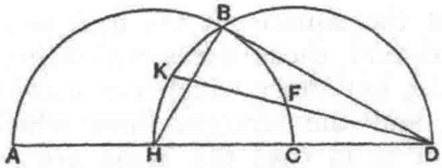
If, says Pappus (IV. p. 178), AB be the rational diameter of a semicircle, and if AB be produced to C so that BC is equal to the radius, if CD be a tangent,



if E be the middle point of the arc BD, and if CE be joined, then CE is the irrational straight line called *minor*. As a matter of fact, if ρ is the radius,

$$CE^2 = \rho^2 (5 - 2\sqrt{3}) \text{ and } CE = \sqrt{\frac{5+\sqrt{13}}{2}} - \sqrt{\frac{5-\sqrt{13}}{2}}.$$

If, again (p. 182), CD be equal to the radius of a semicircle supposed



rational, and if the tangent DB be drawn and the angle ADB be bisected by DF meeting the circumference in F, then DF is the excess by which the *binomial* exceeds the *straight line which produces with a rational area a medial* *whole* (see Eucl. x. 77). (In the figure DK is the *binomial* and KF the other irrational straight line.) As a matter of fact, if ρ be the radius,

$$KD = \rho \cdot \frac{\sqrt{3}+1}{\sqrt{2}}, \text{ and } KF = \rho \cdot \sqrt{\sqrt{3}-1} = \rho \cdot \left(\sqrt{\frac{\sqrt{3}+\sqrt{2}}{2}} - \sqrt{\frac{\sqrt{3}-\sqrt{2}}{2}}\right).$$

Proclus tells us that Euclid left out, as alien to a selection of *elements*, the discussion of the more complicated irrationals, "the unordered irrationals which Apollonius worked out more fully" (Proclus, p. 74, 23), while the scholiast to Book x. remarks that Euclid does not deal with all rationals and irrationals but only the simplest kinds by the combination of which an infinite number of irrationals are obtained, of which Apollonius also gave some. The author of the commentary on Book x. found by Woepcke in an Arabic translation, and above alluded to, also says that "it was Apollonius who, beside the *ordered* irrational magnitudes, showed the existence of the *unordered* and by accurate methods set forth a great number of them." It can only be vaguely gathered, from such hints as the commentator proceeds to give, what the character of the extension of the subject given by Apollonius may have been. See note at end of Book.

DEFINITIONS.

1. Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

2. Straight lines are **commensurable in square** when the squares on them are measured by the same area, and **incommensurable in square** when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called **rational**, and those straight lines which are commensurable with it, whether in length and in square or in square only, **rational**, but those which are incommensurable with it **irrational**.

4. And let the square on the assigned straight line be called **rational** and those areas which are commensurable with it **rational**, but those which are incommensurable with it **irrational**, and the straight lines which produce them **irrational**, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

DEFINITION 1.

Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

DEFINITION 2.

Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ' αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ' αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι.

Commensurable in square is in the Greek δυνάμει σύμμετρος. In earlier translations (e.g. Williamson's) δυνάμει has been translated "in power," but, as the particular *power* represented by δύναμις in Greek geometry is *square*, I have thought it best to use the latter word throughout. It will be observed that Euclid's expression *commensurable in square only* (used in Def. 3 and constantly) corresponds to what Plato makes Theaetetus call a *square root* (δύναμις) in the sense of a *surd*. If a is any straight line, a and $a\sqrt{m}$, or $a\sqrt{m}$ and $a\sqrt{n}$ (where m, n are integers or arithmetical fractions in their lowest terms, proper or improper, but not square) are *commensurable in square only*. Of course (as explained in the Porism to X. 10) all straight lines *commensurable in length* (μήκει), in Euclid's phrase, are commensurable *in square* also; but not all straight lines which are commensurable *in square* are commensurable *in length* as well. On the other hand, straight lines *incommensurable in square* are necessarily incommensurable *in length* also; but not all straight lines which are incommensurable *in length* are incommensurable *in square*. In fact, straight lines which are *commensurable in square only* are incommensurable *in length*, but obviously not incommensurable in square.

DEFINITION 3.

Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλογοι καλείσθωσαν.

The first sentence of the definition is decidedly elliptical. It should, strictly speaking, assert that "with a given straight line there are an infinite number of straight lines which are (1) commensurable either (a) in square only or (b) in square and in length also, and (2) incommensurable, either (a) in length only or (b) in length and in square also."

The relativity of the terms *rational* and *irrational* is well brought out in this definition. We may set out *any straight line* and call it rational, and it is then with reference to this assumed rational straight line that others are called *rational* or *irrational*.

We should carefully note that the signification of *rational* in Euclid is wider than in our terminology. With him, not only is a straight line commensurable *in length* with a rational straight line rational, but a straight line is rational which is commensurable with a rational straight line *in square only*. That is, if ρ is a rational straight line, not only is $\frac{m}{n}\rho$ rational, where m, n are integers and

m/n in its lowest terms is not square, but $\sqrt{\frac{m}{n}}\,.\,\rho$ is *rational* also. We should in this case call $\sqrt{\frac{m}{n}}\,.\,\rho$ irrational. It would appear that Euclid's terminology here differed as much from that of his predecessors as it does from ours. We are familiar with the phrase ἄρρητος διάμετρος τῆς πεμπάδος by which Plato (evidently after the Pythagoreans) describes the diagonal of a square on a straight line containing 5 units of length. This "inexpressible diameter of five (squared)" means $\sqrt{50}$, in contrast to the ῥητὴ διάμετρος, the "expressible diameter" of the same square, by which is meant the approximation $\sqrt{50-1}$, or 7. Thus for Euclid's predecessors $\sqrt{\frac{m}{n}}\,.\,\rho$ would apparently not have been rational but ἄρρητος, "inexpressible," i.e. irrational.

I shall throughout my notes on this Book denote a *rational* straight line in Euclid's sense by ρ, and by ρ and σ when two different rational straight lines are required. Wherever then I use ρ or σ, it must be remembered that ρ, σ may have either of the forms a, $\sqrt{k}\,.\,a$, where a represents a units of length, a being either an integer or of the form m/n, where m, n are both integers, and k is an integer or of the form m/n (where both m, n are integers) but not square. In other words, ρ, σ may have either of the forms a or $\sqrt{A}$, where A represents A units of *area* and A is integral or of the form m/n, where m, n are both integers. It has been the habit of writers to give a and $\sqrt{a}$ as the alternative forms of ρ, but I shall always use $\sqrt{A}$ for the second in order to keep the dimensions right, because it must be borne in mind throughout that ρ is an irrational *straight line.*

As Euclid extends the signification of *rational* (ῥητός, literally *expressible*), so he limits the scope of the term ἄλογος (literally *having no ratio*) as applied to straight lines. That this limitation was started by himself may perhaps be inferred from the form of words "*let* straight lines incommensurable with it *be called* irrational." Irrational straight lines then are with Euclid straight lines commensurable *neither in length nor in square* with the assumed rational straight line. $\sqrt{k}\,.\,a$ where k is not square is not irrational; $\sqrt[4]{k}\,.\,a$ is irrational, and so (as we shall see later on) is $(\sqrt{k} \pm \sqrt{\lambda})\,a$.

DEFINITION 4.

Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

As applied to *areas*, the terms *rational* and *irrational* have, on the other hand, the same sense with Euclid as we should attach to them. According to Euclid, if ρ is a rational straight line in *his* sense, ρ^2 is *rational* and any area commensurable with it, i.e. of the form $k\rho^2$ (where k is an integer, or of the form m/n, where m, n are integers), is rational; but any area of the form $\sqrt{k}\,.\,\rho^2$ is *irrational.* Euclid's *rational area* thus contains *A units of area*, where A is an integer or of the form m/n, where m, n are integers; and his *irrational area* is of the form $\sqrt{k}\,.\,A$. His irrational *area* is then connected with his irrational *straight line* by making the latter the square root of the former. This would give us for the irrational *straight line* $\sqrt[4]{k} . \sqrt{A}$, which of course includes $\sqrt[4]{k} . a$.

αἱ δυνάμεναι αὐτά are the straight lines the squares on which are equal to the areas, in accordance with the regular meaning of δύνασθαι. It is scarcely possible, in a book written in geometrical language, to translate δυναμένη as the *square root* (of an *area*) and δύνασθαι as *to be the square root* (of an *area*), although I can use the term "square root" when in my notes I am using an algebraical expression to represent an area; I shall therefore hereafter use the word "side" for δυναμένη and "to be the side of" for δύνασθαι, so that "side" will in such expressions be a short way of expressing the "side of a *square equal to* (*an area*)." In this particular passage it is not quite practicable to use the words "side of" or "straight line the square on which is equal to," for these expressions occur just afterwards for two alternatives which the word δυναμένη covers. I have therefore exceptionally translated "the straight lines which produce them" (i.e. if squares are described upon them as sides).

αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι, literally "the (straight lines) which *describe* squares equal to them": a peculiar use of the active of ἀναγράφειν, the meaning being of course "the straight lines on which *are described* the squares" which are equal to the rectilineal figures.