

M. Loève

Probability Theory I

4th Edition

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PREFACE TO THE FOURTH EDITION

This fourth edition contains several additions. The main ones concern three closely related topics: Brownian motion, functional limit distributions, and random walks. Besides the power and ingenuity of their methods and the depth and beauty of their results, their importance is fast growing in Analysis as well as in theoretical and applied Probability:

These additions increased the book to an unwieldy size and it had to be split into two volumes.

About half of the first volume is devoted to an elementary introduction, then to mathematical foundations and basic probability concepts and tools. The second half is devoted to a detailed study of Independence which played and continues to play a central role both by itself and as a catalyst.

The main additions consist of a section on convergence of probabilities on metric spaces and a chapter whose first section on domains of attraction completes the study of the Central limit problem, while the second one is devoted to random walks.

About a third of the second volume is devoted to conditioning and properties of sequences of various types of dependence. The other two thirds are devoted to random functions; the last Part on Elements of random analysis is more sophisticated.

The main addition consists of a chapter on Brownian motion and limit distributions.

It is strongly recommended that the reader begin with less involved portions. In particular, the starred ones ought to be left out until they are needed or unless the reader is especially interested in them.

I take this opportunity to thank Mrs. Rubalcava for her beautiful typing of all the editions since the inception of the book. I also wish to thank the editors of Springer-Verlag, New York, for their patience and care.

M. L.

*January, 1977
Berkeley, California*

PREFACE TO THE THIRD EDITION

This book is intended as a text for graduate students and as a reference for workers in Probability and Statistics. The prerequisite is honest calculus. The material covered in Parts Two to Five inclusive requires about three to four semesters of graduate study. The introductory part may serve as a text for an undergraduate course in elementary probability theory.

The Foundations are presented in:

the Introductory Part on the background of the concepts and problems, treated without advanced mathematical tools;

Part One on the Notions of Measure Theory that every probabilist and statistician requires;

Part Two on General Concepts and Tools of Probability Theory.

Random sequences whose general properties are given in the Foundations are studied in:

Part Three on Independence devoted essentially to sums of independent random variables and their limit properties;

Part Four on Dependence devoted to the operation of conditioning and limit properties of sums of dependent random variables. The last section introduces random functions of second order.

Random functions and processes are discussed in:

Part Five on Elements of random analysis devoted to the basic concepts of random analysis and to the martingale, decomposable, and Markov types of random functions.

Since the primary purpose of the book is didactic, methods are emphasized and the book is subdivided into:

unstarred portions, independent of the remainder; starred portions, which are more involved or more abstract;

complements and details, including illustrations and applications of the material in the text, which consist of propositions with fre-

PREFACE TO THE THIRD EDITION

quent hints; most of these propositions can be found in the articles and books referred to in the Bibliography.

Also, for teaching and reference purposes, it has proved useful to name most of the results.

Numerous historical remarks about results, methods, and the evolution of various fields are an intrinsic part of the text. The purpose is purely didactic: to attract attention to the basic contributions while introducing the ideas explored. Books and memoirs of authors whose contributions are referred to and discussed are cited in the Bibliography, which parallels the text in that it is organized by parts and, within parts, by chapters. Thus the interested student can pursue his study in the original literature.

This work owes much to the reactions of the students on whom it has been tried year after year. However, the book is definitely more concise than the lectures, and the reader will have to be armed permanently with patience, pen, and calculus. Besides, in mathematics, as in any form of poetry, the reader has to be a poet *in posse*.

This third edition differs from the second (1960) in a number of places. Modifications vary all the way from a prefix ("sub" martingale in lieu of "semi"-martingale) to an entire subsection (§36.2). To preserve pagination, some additions to the text proper (especially 9, p. 656) had to be put in the Complements and Details. It is hoped that moreover most of the errors have been eliminated and that readers will be kind enough to inform the author of those which remain.

I take this opportunity to thank those whose comments and criticisms led to corrections and improvements: for the first edition, E. Barankin, S. Bochner, E. Parzen, and H. Robbins; for the second edition, Y. S. Chow, R. Cogburn, J. L. Doob, J. Feldman, B. Jamison, J. Karush, P. A. Meyer, J. W. Pratt, B. A. Sevastianov, J. W. Woll; for the third edition, S. Dharmadhikari, J. Fabius, D. Freedman, A. Maitra, U. V. Prokhorov. My warm thanks go to Cogburn, whose constant help throughout the preparation of the second edition has been invaluable. This edition has been prepared with the partial support of the Office of Naval Research and of the National Science Foundation.

M. L.

April, 1962
Berkeley, California

CONTENTS OF VOLUME II

GRADUATE TEXTS IN MATHEMATICS VOL. 46

DEPENDENCE

CONDITIONING

- Concept of Conditioning
- Properties of Conditioning
- Regular Pr. Functions
- Conditional Distributions

FROM INDEPENDENCE TO DEPENDENCE

- Central Asymptotic Problem
- Centerings, Martingales, and a.s. Convergence

ERGODIC THEOREMS

- Translation of Sequences; Basic Ergodic Theorem and Stationarity
- Ergodic Theorems and L_r -Spaces
- Ergodic Theorems on Banach Spaces

SECOND ORDER PROPERTIES

- Orthogonality
- Second Order Random Functions

ELEMENTS OF RANDOM ANALYSIS

FOUNDATIONS; MARTINGALES AND DECOMPOSABILITY

- Foundations
- Martingales
- Decomposability

BROWNIAN MOTION AND LIMIT DISTRIBUTIONS

- Brownian Motion
- Limit Distributions

MARKOV PROCESSES

Markov Dependence

Time-Continuous Transition Probabilities

Markov Semi-Groups

Sample Continuity and Diffusion Operators

BIBLIOGRAPHY**INDEX**

CONTENTS OF VOLUME I

GRADUATE TEXTS IN MATHEMATICS VOL. 45

INTRODUCTORY PART: ELEMENTARY PROBABILITY THEORY

SECTION	PAGE
I. INTUITIVE BACKGROUND	3
1. Events	3
2. Random events and trials	5
3. Random variables	6
II. AXIOMS; INDEPENDENCE AND THE BERNOULLI CASE	8
1. Axioms of the finite case	8
2. Simple random variables	9
3. Independence	11
4. Bernoulli case	12
5. Axioms for the countable case	15
6. Elementary random variables	17
7. Need for nonelementary random variables	22
III. DEPENDENCE AND CHAINS	24
1. Conditional probabilities	24
2. Asymptotically Bernoullian case	25
3. Recurrence	26
4. Chain dependence	28
*5. Types of states and asymptotic behavior	30
*6. Motion of the system	36
*7. Stationary chains	39
COMPLEMENTS AND DETAILS	42

PART ONE: NOTIONS OF MEASURE THEORY

CHAPTER I: SETS, SPACES, AND MEASURES

1. SETS, CLASSES, AND FUNCTIONS	55
1.1 Definitions and notations	55
1.2 Differences, unions, and intersections	56
1.3 Sequences and limits	57
1.4 Indicators of sets	59

SECTION	PAGE
1.5 Fields and σ -fields	59
1.6 Monotone classes	60
*1.7 Product sets	61
*1.8 Functions and inverse functions	62
*1.9 Measurable spaces and functions	64
*2. TOPOLOGICAL SPACES	65
*2.1 Topologies and limits	66
*2.2 Limit points and compact spaces	69
*2.3 Countability and metric spaces	72
*2.4 Linearity and normed spaces	78
3. ADDITIVE SET FUNCTIONS	83
3.1 Additivity and continuity	83
3.2 Decomposition of additive set functions	87
*4. CONSTRUCTION OF MEASURES ON σ -FIELDS	88
*4.1 Extension of measures	88
*4.2 Product probabilities	91
*4.3 Consistent probabilities on Borel fields	93
*4.4 Lebesgue-Stieltjes measures and distribution functions	96
COMPLEMENTS AND DETAILS	100
CHAPTER II: MEASURABLE FUNCTIONS AND INTEGRATION	
5. MEASURABLE FUNCTIONS	103
5.1 Numbers	103
5.2 Numerical functions	105
5.3 Measurable functions	107
6. MEASURE AND CONVERGENCES	111
6.1 Definitions and general properties	111
6.2 Convergence almost everywhere	114
6.3 Convergence in measure	116
7. INTEGRATION	118
7.1 Integrals	119
7.2 Convergence theorems	125
8. INDEFINITE INTEGRALS; ITERATED INTEGRALS	130
8.1 Indefinite integrals and Lebesgue decomposition	130
8.2 Product measures and iterated integrals	135
*8.3 Iterated integrals and infinite product spaces	137
COMPLEMENTS AND DETAILS	139

SECTION

PAGE

PART TWO: GENERAL CONCEPTS AND TOOLS OF PROBABILITY THEORY

CHAPTER III: PROBABILITY CONCEPTS

9. PROBABILITY SPACES AND RANDOM VARIABLES	151
9.1 Probability terminology	151
*9.2 Random vectors, sequences, and functions	155
9.3 Moments, inequalities, and convergences	156
*9.4 Spaces L_r	162
10. PROBABILITY DISTRIBUTIONS	168
10.1 Distributions and distribution functions	168
10.2 The essential feature of pr. theory	172
COMPLEMENTS AND DETAILS	174

CHAPTER IV: DISTRIBUTION FUNCTIONS AND CHARACTERISTIC FUNCTIONS

11. DISTRIBUTION FUNCTIONS	177
11.1 Decomposition	177
11.2 Convergence of d.f.'s	180
11.3 Convergence of sequences of integrals	182
*11.4 Further extension and convergence of moments	184
*11.5 Discussion	187
*12. CONVERGENCE OF PROBABILITIES ON METRIC SPACES	189
*12.1 Convergence	190
*12.2 Regularity and tightness	193
*12.3 Tightness and relative compactness	195
13. CHARACTERISTIC FUNCTIONS AND DISTRIBUTION FUNCTIONS	198
13.1 Uniqueness	199
13.2 Convergences	202
13.3 Composition of d.f.'s and multiplication of ch.f.'s	206
13.4 Elementary properties of ch.f.'s and first applications	207
14. PROBABILITY LAWS AND TYPES OF LAWS	214
14.1 Laws and types; the degenerate type	214
14.2 Convergence of types	216
14.3 Extensions	218
15. NONNEGATIVE-DEFINITENESS; REGULARITY	218
15.1 Ch.f.'s and nonnegative-definiteness	218
*15.2 Regularity and extension of ch.f.'s	223

SECTION	PAGE
*15.3 Composition and decomposition of regular ch.f.'s	226
COMPLEMENTS AND DETAILS	227

PART THREE: INDEPENDENCE

CHAPTER V: SUMS OF INDEPENDENT RANDOM VARIABLES

16. CONCEPT OF INDEPENDENCE	235
16.1 Independent classes and independent functions	235
16.2 Multiplication properties	238
16.3 Sequences of independent r.v.'s	240
*16.4 Independent r.v.'s and product spaces	242
17. CONVERGENCE AND STABILITY OF SUMS; CENTERING AT EXPECTATIONS AND TRUNCATION	243
17.1 Centering at expectations and truncation	244
17.2 Bounds in terms of variances	246
17.3 Convergence and stability	248
*17.4 Generalization	252
*18. CONVERGENCE AND STABILITY OF SUMS; CENTERING AT MEDIANs AND SYMMETRIZATION	255
*18.1 Centering at medians and symmetrization	256
*18.2 Convergence and stability	260
*19. EXPONENTIAL BOUNDS AND NORMED SUMS	266
*19.1 Exponential bounds	266
*19.2 Stability	270
*19.3 Law of the iterated logarithm	272
COMPLEMENTS AND DETAILS	275

CHAPTER VI: CENTRAL LIMIT PROBLEM

20. DEGENERATE, NORMAL, AND POISSON TYPES	280
20.1 First limit theorems and limit laws	280
*20.2 Composition and decomposition	283
21. EVOLUTION OF THE PROBLEM	286
21.1 The problem and preliminary solutions	286
21.2 Solution of the Classical Limit Problem	290
*21.3 Normal approximation	294

SECTION	PAGE
22. CENTRAL LIMIT PROBLEM; THE CASE OF BOUNDED VARIANCES	300
22.1 Evolution of the problem	300
22.2 The case of bounded variances	302
*23. SOLUTION OF THE CENTRAL LIMIT PROBLEM	308
*23.1 A family of limit laws; the infinitely decomposable laws	308
*23.2 The uan condition	314
*23.3 Central Limit Theorem	319
*23.4 Central convergence criterion	323
23.5 Normal, Poisson, and degenerate convergence	327
*24. NORMED SUMS	331
*24.1 The problem	331
*24.2 Norming sequences	332
*24.3 Characterization of \mathfrak{N}	334
*24.4 Identically distributed summands and stable laws	338
24.5 Lévy representation	343
COMPLEMENTS AND DETAILS	349

CHAPTER VII: INDEPENDENT IDENTICALLY DISTRIBUTED SUMMANDS

25. REGULAR VARIATION AND DOMAINS OF ATTRACTION	353
25.1 Regular variation	354
25.2 Domains of attraction	360
26. RANDOM WALK	368
26.1 Set-up and basic implications	369
26.2 Dichotomy: recurrence and transience	381
26.3 Fluctuations; exponential identities	388
26.4 Fluctuations; asymptotic behaviour	396
COMPLEMENTS AND DETAILS	401
BIBLIOGRAPHY	407
INDEX	413

Introductory Part

ELEMENTARY PROBABILITY THEORY

Probability theory is concerned with the mathematical analysis of the intuitive notion of "chance" or "randomness," which, like all notions, is born of experience. The quantitative idea of randomness first took form at the gaming tables, and probability theory began, with Pascal and Fermat (1654), as a theory of games of chance. Since then, the notion of chance has found its way into almost all branches of knowledge. In particular, the discovery that physical "observables," even those which describe the behavior of elementary particles, were to be considered as subject to laws of chance made an investigation of the notion of chance basic to the whole problem of rational interpretation of nature.

A theory becomes mathematical when it sets up a mathematical model of the phenomena with which it is concerned, that is, when, to describe the phenomena, it uses a collection of well-defined symbols and operations on the symbols. As the number of phenomena, together with their known properties, increases, the mathematical model evolves from the early crude notions upon which our intuition was built in the direction of higher generality and abstractness.

In this manner, the inner consistency of the model of random phenomena became doubtful, and this forced a rebuilding of the whole structure in the second quarter of this century, starting with a formulation in terms of axioms and definitions. Thus, there appeared a branch of pure mathematics—probability theory—concerned with the construction and investigation *per se* of the mathematical model of randomness.

The purpose of the Introductory Part (of which the other parts of this book are independent) is to give "intuitive meaning" to the concepts and problems of probability theory. First, by analyzing briefly

some ideas derived from everyday experience—especially from games of chance—we shall arrive at an elementary axiomatic setup; we leave the illustrations with coins, dice, cards, darts, etc., to the reader. Then, we shall apply this axiomatic setup to describe in a precise manner and to investigate in a rigorous fashion a few of the “intuitive notions” relative to randomness. No special tools will be needed, whereas in the nonelementary setup measure-theoretic concepts and Fourier-Stieltjes transforms play a prominent rôle.

I. INTUITIVE BACKGROUND

1. **Events.** The primary notion in the understanding of nature is that of *event*—the occurrence or nonoccurrence of a phenomenon. The abstract concept of event pertains only to its occurrence or nonoccurrence and not to its nature. This is the concept we intend to analyze. We shall denote events by A, B, C, \dots with or without affixes.

To every event A there corresponds a contrary event “not A ,” to be denoted by A^c ; A^c occurs if, and only if, A does not occur. An event may imply another event: A *implies* B if, when A occurs, then B necessarily occurs; we write $A \subset B$. If A implies B and also B implies A , then we say that A and B are *equivalent*; we write $A = B$. The nature of two equivalent events may be different, but as long as we are concerned only with occurrence or nonoccurrence, they can and will be identified. Events are combined into new events by means of operations expressed by the terms “and,” “or” and “not.”

A “and” B is an event which occurs if, and only if, both the event A and the event B occur; we denote it by $A \cap B$ or, simply, by AB . If AB cannot occur (that is, if A occurs, then B does not occur, and if B occurs, then A does not occur), we say that the event A and the event B are *disjoint* (exclude one another, are mutually exclusive, are incompatible).

A “or” B is an event which occurs if, and only if, at least one of the events A, B occurs; we denote it by $A \cup B$. If, and only if, A and B are disjoint, we replace “or” by $+$. Similarly, more than two events can be combined by means of “and,” “or”; we write

$$A_1 \cap A_2 \cap \dots \cap A_n \quad \text{or} \quad A_1 A_2 \dots A_n \quad \text{or} \quad \bigcap_{k=1}^n A_k,$$

$$A_1 \cup A_2 \cup \dots \cup A_n \quad \text{or} \quad \bigcup_{k=1}^n A_k, \quad A_1 + A_2 + \dots + A_n \quad \text{or} \quad \sum_{k=1}^n A_k.$$

There are two combinations of events which can be considered as “boundary events”; they are the first and the last events—in terms of

implication. Events of the form $A + A^c$ can be said to represent an “always occurrence,” for they can only occur. Since, whatever be the event A , the events $A + A^c$ and the events they imply are equivalent, all such events are to be identified and will be called the *sure event*, to be denoted by Ω . Similarly, events of the form AA^c and the events which imply them, which can be said to represent a “never occurrence” for they cannot occur, are to be identified, and will be called the *impossible event*, to be denoted by \emptyset ; thus, the definition of disjoint events A and B can be written $AB = \emptyset$. The impossible and the sure events are “first” and “last” events, for, whatever be the event A , we have $\emptyset \subset A \subset \Omega$.

The interpretation of symbols \subset , $=$, \cap , \cup , in terms of occurrence and nonoccurrence, shows at once that

if $A \subset B$, then $B^c \subset A^c$, and conversely;

$$AB = BA, \quad A \cup B = B \cup A;$$

$$(AB)C = A(BC), \quad (A \cup B) \cup C = A \cup (B \cup C);$$

$$A(B \cup C) = AB \cup AC, \quad A \cup BC = (A \cup B)(A \cup C);$$

$$(AB)^c = A^c \cup B^c, \quad (A \cup B)^c = A^c B^c, \quad A \cup B = A + A^c B;$$

more generally

$$\left(\bigcap_{k=1}^n A_k\right)^c = \bigcup_{k=1}^n A_k^c, \quad \left(\bigcup_{k=1}^n A_k\right)^c = \bigcap_{k=1}^n A_k^c,$$

and so on:

We recognize here the rules of operations on sets. In terms of sets, Ω is the space in which lie the sets A, B, C, \dots , \emptyset is the *empty set*, A^c is the set *complementary* to the set A ; AB is the *intersection*, $A \cup B$ is the *union* of the sets A and B , and $A \subset B$ means that A is contained in B .

In science, or, more precisely, in the investigation of “laws of nature,” events are classified into conditions and outcomes of an experiment. *Conditions* of an experiment are events which are known or are made to occur. *Outcomes* of an experiment are events which *may* occur when the experiment is performed, that is, when its conditions occur. All (finite) combinations of outcomes by means of “not,” “and,” “or,” are outcomes; in the terminology of sets, the outcomes of an experiment form a *field* (or an “algebra” of sets). The conditions of an experiment,