M. Loève

Probability Theory I

4th Edition

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Springer-Verlag

New York Heidelberg Berlin

M. Loève
Departments of Mathematics and Statistics
University of California at Berkeley
Berkeley, California 94720

AMS Subject Classifications
28-01, 60A05, 60Bxx, 60E05, 60Fxx

Library of Congress Cataloging in Publication Data

Loève, Michel, 1907–
Probability theory.
(Graduate texts in mathematics; 45)
Bibliography p.
Includes index.
1. Probabilities. I. Title. II. Series.
QA273.L63 1977 519.2 76-28332

Originally published in the University Series in Higher Mathematics (D. Van Nostrand Company); edited by M. H. Stone, L. Nirenberg, and S. S. Chern.

Printed in the United States of America.

ISBN 0-387-90210-4 Springer-Verlag New York
ISBN 3-540-90210-4 Springer-Verlag Berlin Heidelberg

PREFACE TO THE FOURTH EDITION

This fourth edition contains several additions. The main ones concern three closely related topics: Brownian motion, functional limit distributions, and random walks. Besides the power and ingenuity of their methods and the depth and beauty of their results, their importance is fast growing in Analysis as well as in theoretical and applied Probability.

These additions increased the book to an unwieldy size and it had to be split into two volumes.

About half of the first volume is devoted to an elementary introduction, then to mathematical foundations and basic probability concepts and tools. The second half is devoted to a detailed study of Independence which played and continues to play a central role both by itself and as a catalyst.

The main additions consist of a section on convergence of probabilities on metric spaces and a chapter whose first section on domains of attraction completes the study of the Central limit problem, while the second one is devoted to random walks.

About a third of the second volume is devoted to conditioning and properties of sequences of various types of dependence. The other two thirds are devoted to random functions; the last Part on Elements of random analysis is more sophisticated.

The main addition consists of a chapter on Brownian motion and limit distributions.

It is strongly recommended that the reader begin with less involved portions. In particular, the starred ones ought to be left out until they are needed or unless the reader is especially interested in them.

I take this opportunity to thank Mrs. Rubalcava for her beautiful typing of all the editions since the inception of the book. I also wish to thank the editors of Springer-Verlag, New York, for their patience and care.

M. L.

January, 1977
Berkeley, California

PREFACE TO THE THIRD EDITION

This book is intended as a text for graduate students and as a reference for workers in Probability and Statistics. The prerequisite is honest calculus. The material covered in Parts Two to Five inclusive requires about three to four semesters of graduate study. The introductory part may serve as a text for an undergraduate course in elementary probability theory.

The Foundations are presented in:

the Introductory Part on the background of the concepts and problems, treated without advanced mathematical tools;
Part One on the Notions of Measure Theory that every probabilist and statistician requires;
Part Two on General Concepts and Tools of Probability Theory.

Random sequences whose general properties are given in the Foundations are studied in:

Part Three on Independence devoted essentially to sums of independent random variables and their limit properties;
Part Four on Dependence devoted to the operation of conditioning and limit properties of sums of dependent random variables. The last section introduces random functions of second order.

Random functions and processes are discussed in:

Part Five on Elements of random analysis devoted to the basic concepts of random analysis and to the martingale, decomposable, and Markov types of random functions.

Since the primary purpose of the book is didactic, methods are emphasized and the book is subdivided into:

unstarred portions, independent of the remainder; starred portions, which are more involved or more abstract;
complements and details, including illustrations and applications of the material in the text, which consist of propositions with fre-

quent hints; most of these propositions can be found in the articles and books referred to in the Bibliography.

Also, for teaching and reference purposes, it has proved useful to name most of the results.

Numerous historical remarks about results, methods, and the evolution of various fields are an intrinsic part of the text. The purpose is purely didactic: to attract attention to the basic contributions while introducing the ideas explored. Books and memoirs of authors whose contributions are referred to and discussed are cited in the Bibliography, which parallels the text in that it is organized by parts and, within parts, by chapters. Thus the interested student can pursue his study in the original literature.

This work owes much to the reactions of the students on whom it has been tried year after year. However, the book is definitely more concise than the lectures, and the reader will have to be armed permanently with patience, pen, and calculus. Besides, in mathematics, as in any form of poetry, the reader has to be a poet *in posse*.

This third edition differs from the second (1960) in a number of places. Modifications vary all the way from a prefix ("sub" martingale in lieu of "semi"-martingale) to an entire subsection (§36.2). To preserve pagination, some additions to the text proper (especially 9, p. 656) had to be put in the Complements and Details. It is hoped that moreover most of the errors have been eliminated and that readers will be kind enough to inform the author of those which remain.

I take this opportunity to thank those whose comments and criticisms led to corrections and improvements: for the first edition, E. Barankin, S. Bochner, E. Parzen, and H. Robbins; for the second edition, Y. S. Chow, R. Cogburn, J. L. Doob, J. Feldman, B. Jamison, J. Karush, P. A. Meyer, J. W. Pratt, B. A. Sevastianov, J. W. Woll; for the third edition, S. Dharmadhikari, J. Fabius, D. Freedman, A. Maitra, U. V. Prokhorov. My warm thanks go to Cogburn, whose constant help throughout the preparation of the second edition has been invaluable. This edition has been prepared with the partial support of the Office of Naval Research and of the National Science Foundation.

M. L.

April, 1962
Berkeley, California

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Introductory Part

ELEMENTARY PROBABILITY THEORY

Probability theory is concerned with the mathematical analysis of the intuitive notion of "chance" or "randomness," which, like all notions, is born of experience. The quantitative idea of randomness first took form at the gaming tables, and probability theory began, with Pascal and Fermat (1654), as a theory of games of chance. Since then, the notion of chance has found its way into almost all branches of knowledge. In particular, the discovery that physical "observables," even those which describe the behavior of elementary particles, were to be considered as subject to laws of chance made an investigation of the notion of chance basic to the whole problem of rational interpretation of nature.

A theory becomes mathematical when it sets up a mathematical model of the phenomena with which it is concerned, that is, when, to describe the phenomena, it uses a collection of well-defined symbols and operations on the symbols. As the number of phenomena, together with their known properties, increases, the mathematical model evolves from the early crude notions upon which our intuition was built in the direction of higher generality and abstractness.

In this manner, the inner consistency of the model of random phenomena became doubtful, and this forced a rebuilding of the whole structure in the second quarter of this century, starting with a formulation in terms of axioms and definitions. Thus, there appeared a branch of pure mathematics—probability theory—concerned with the construction and investigation *per se* of the mathematical model of randomness.

The purpose of the Introductory Part (of which the other parts of this book are independent) is to give "intuitive meaning" to the concepts and problems of probability theory. First, by analyzing briefly some ideas derived from everyday experience—especially from games of chance—we shall arrive at an elementary axiomatic setup; we leave the illustrations with coins, dice, cards, darts, etc., to the reader. Then, we shall apply this axiomatic setup to describe in a precise manner and to investigate in a rigorous fashion a few of the "intuitive notions" relative to randomness. No special tools will be needed, whereas in the nonelementary setup measure-theoretic concepts and Fourier-Stieltjes transforms play a prominent role.

I. INTUITIVE BACKGROUND

1. Events. The primary notion in the understanding of nature is that of *event*—the occurrence or nonoccurrence of a phenomenon. The abstract concept of event pertains only to its occurrence or nonoccurrence and not to its nature. This is the concept we intend to analyze. We shall denote events by $A, B, C, \cdots$ with or without affixes.

To every event A there corresponds a contrary event "not A," to be denoted by A^c; A^c occurs if, and only if, A does not occur. An event may imply another event: *A implies B* if, when A occurs, then B necessarily occurs; we write $A \subset B$. If A implies B and also B implies A, then we say that A and B are *equivalent;* we write $A = B$. The nature of two equivalent events may be different, but as long as we are concerned only with occurrence or nonoccurrence, they can and will be identified. Events are combined into new events by means of operations expressed by the terms "and," "or" and "not."

A "*and*" B is an event which occurs if, and only if, both the event A and the event B occur; we denote it by $A \cap B$ or, simply, by AB. If AB cannot occur (that is, if A occurs, then B does not occur, and if B occurs, then A does not occur), we say that the event A and the event B are *disjoint* (exclude one another, are mutually exclusive, are incompatible).

A "*or*" B is an event which occurs if, and only if, at least one of the events A, B occurs; we denote it by $A \cup B$. If, and only if, A and B are disjoint, we replace "or" by $+$. Similarly, more than two events can be combined by means of "and," "or"; we write

$$A_1 \cap A_2 \cap \cdots \cap A_n \quad \text{or} \quad A_1 A_2 \cdots A_n \quad \text{or} \quad \bigcap_{k=1}^{n} A_k,$$

$$A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{or} \quad \bigcup_{k=1}^{n} A_k, \quad A_1 + A_2 + \cdots A_n \quad \text{or} \quad \sum_{k=1}^{n} A_k.$$

There are two combinations of events which can be considered as "boundary events"; they are the first and the last events—in terms of implication. Events of the form $A + A^c$ can be said to represent an "always occurrence," for they can only occur. Since, whatever be the event A, the events $A + A^c$ and the events they imply are equivalent, all such events are to be identified and will be called the *sure event*, to be denoted by Ω. Similarly, events of the form AA^c and the events which imply them, which can be said to represent a "never occurrence" for they cannot occur, are to be identified, and will be called the *impossible event*, to be denoted by $\emptyset$; thus, the definition of disjoint events A and B can be written $AB = \emptyset$. The impossible and the sure events are "first" and "last" events, for, whatever be the event A, we have $\emptyset \subset A \subset \Omega$.

The interpretation of symbols $\subset$, $=$, $\cap$, $\cup$, in terms of occurrence and nonoccurrence, shows at once that

$$\text{if} \quad A \subset B, \quad \text{then} \quad B^c \subset A^c, \quad \text{and conversely;}$$

$$AB = BA, \quad A \cup B = B \cup A;$$

$$(AB)C = A(BC), \quad (A \cup B) \cup C = A \cup (B \cup C);$$

$$A(B \cup C) = AB \cup AC, \quad A \cup BC = (A \cup B)(A \cup C);$$

$$(AB)^c = A^c \cup B^c, \quad (A \cup B)^c = A^cB^c, \quad A \cup B = A + A^cB;$$

more generally

$$\Big(\bigcap_{k=1}^{n} A_k\Big)^c = \bigcup_{k=1}^{n} A_k^c, \quad \Big(\bigcup_{k=1}^{n} A_k\Big)^c = \bigcap_{k=1}^{n} A_k^c,$$

and so on.

We recognize here the rules of operations on sets. In terms of sets, Ω is the space in which lie the sets $A, B, C, \cdots, \emptyset$ is the *empty* set, A^c is the set *complementary* to the set A; AB is the *intersection*, $A \cup B$ is the *union* of the sets A and B, and $A \subset B$ means that A is contained in B.

In science, or, more precisely, in the investigation of "laws of nature," events are classified into conditions and outcomes of an experiment. *Conditions* of an experiment are events which are known or are made to occur. *Outcomes* of an experiment are events which *may* occur when the experiment is performed, that is, when its conditions occur. All (finite) combinations of outcomes by means of "not," "and," "or," are outcomes; in the terminology of sets, the outcomes of an experiment form a *field* (or an "algebra" of sets). The conditions of an experiment,