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VOLUME II

Communication, Control, Devices, and Systems

Edited by

SHELDON S. L. CHANG,

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PREFACE

The three volumes of this "fundamentals handbook" are designed to meet a need frequently felt by engineers: the need to broaden themselves and to keep up with technological developments so that they can do a better job. In an environment of rapid technological and scientific advance, the importance of keeping up-to-date is self-evident. The importance of not overspecializing is less recognized, but it is there. A versatile engineer is a tremendous asset to an employer in meeting today's changing needs, ways, and means.

These three volumes constitute a coherent, concise treatise of the core areas of electrical and computer engineering with emphasis on system, device, and circuit design. While a handbook of this type is not new in other disciplines—consider *The Handbook of Physics*, for example—it is the first of its kind in electrical engineering. Each volume of *The Handbook of Physics* is devoted to a significant area of research, and a similar handbook of electrical engineering at the same level would be enormous. However, the engineer's job, unlike the physicist's is not to search for new engineering principles but to apply existing principles to design and develop new products. The level of these three volumes was determined with this objective in mind.

Volume II covers engineering systems: communication, control, instrumentation, power. and energy. Section 1 gives a brief, self-contained presentation of statistical methods, including some aspects of statistical physics used in engineering systems analysis. In line with current literature, probability is introduced as a "measure" on elementary events. However, the meaning of "measure" is explained heuristically and no previous knowledge of measure theory is assumed. Among the topics covered, stochastic processes, queueing theory, and thermal noise are useful in the analysis and design of communication, control, and instrumentation systems, while Maxwell distribution and transport processes are essential to the understanding of energy devices. Section 2, on communication principles, presents basic information for designing communication systems: the time and frequency uncertainty relationship, the sampling theorem, modulation and noise, coherent detection, Bayesian decision, information theory and coding, baseband shaping, optimum performance of communication systems, equalizers, multiplexing, and spread spectrum communication. Section 3 starts with an analytical treatment of communication electronics for AM and FM systems and then proceeds to two major application areas, satellite communication and radar. The satellite communication topics covered are system engineering, hardware, multiple access techniques, capacity allocation, packet switching, and satellite-aided mobile radio systems. The material on radar technology includes principles as well as advances and limitations in the modes of radar operation, data analysis, and presentation. Section 4 joins the concepts of classical and modern optimal control and presents the material so that it is accessible to design and project engineers. Optimal linear control and filtering is derived by cost minimization (simply completing the squares). An extensive set of simple computer programs for designing optimal control and filtering and simulation in both continuous and discrete time is provided to aid the practicing engineer.

Section 5 starts with an overview of instruments and then proceeds to commonly used subsystems, sensors and transducers, instrumentation amplifiers, microprocessors in instrumentation, and output devices including oscilloscopes, digital multimeters, and so on. Each of the instrument subsystems is discussed in detail with emphasis given to salient design considerations. Instruments in two specialized areas, biomedical instrumentation and telecommunication and telemetry instrumentation, are given selected coverage.

Section 6, on power apparatus and systems, gives thorough coverage in selected areas: motors and generators are analyzed with the equivalent circuit approach in which the power absorbed or delivered from a virtual element represents conversion. While it is the simplest approach, it is also *the* approach used by design engineers to evaluate the effects on performance of machine parameters, magnetic saturation, harmonics, and so on. Sections on transmission lines, carrier communication, and relaying present important design information and are written carefully, without assuming previous knowledge of the subjects. An article on economic and environmental issues of power generation is also included.

Section 7, on energy engineering, covers energy sources and conversion technologies that are likely to have maximum impact in the coming decade: solar energy conservation and utilization, solar cells, thermionic and MHD converters, and nuclear power reactors. Also included is a short article on energy policy analysis using the Brookhaven energy reference system.

I am very grateful to Drs. J. G. Truxal, R. W. Licky, R. E. Miller, and S. S. Director for their many helpful suggestions, to Da Zhong Zheng, Dao Rung Hsu, and Wen Min Pan, Visiting Scholars from the People's Republic of China, for their help and helpful suggestions in the final preparation of the manuscript, and to the section editors and contributors for their help. Many of the section editors and contributors are leaders in their respective areas of specialization, and their reputations are far beyond what these three volumes can hope to enhance. They contributed because they share a common goal: to help our fellow engineers help themselves and to share American know-how with the rest of the world in a most effective way.

SHELDON S. L. CHANG

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CONTENTS

SECTION 1

PROBABILITY, STATISTICS, STOCHASTIC AND QUEUEING PROCESSES, STATISTICAL PHYSICS

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The present section gives a brief self-contained introduction of the subjects listed. Although each subject is an important field of study by itself, there are the same sets of ideas and methodology which make a unified presentation effective.

Sections 1.1 and 1.2 on probability and statistical distributions give the basic mathematical concepts and methodology. Probability and conditional probability are defined in terms of measures on a point set. Bayes' theorem follows directly from the definitions. The concepts of a random variable, mean value, standard deviation, and variances are introduced which lead to the central limit theorem, moment-generating function, and various statistical distributions.

Section 1.3 gives a brief introduction to elements of queueing theory which have found many applications in communication and computer engineering. Equations for the $M/M/1$ queue, $M/M/n$ queue, and Erlang distribution are derived. A problem that cannot be tackled with steady-state queueing theory but in existence everywhere is that of queueing dynamics. An approximate solution is given in terms of two differential equations for the mean value and variance of queue length. These equations can be numerically integrated to predict queueing behavior under transient and time-varying conditions.

Section 1.4 on stochastic processes introduces the concepts of stationarity, correlation, and spectral density functions. Wiener theorem relates correlation function, and spectral density as transform pairs of each other. Transmission equations that relate the statistical properties of the input and output variables of a linear system are derived. Gaussian distribution, white noise processes, and their applications to communication and control problems are discussed.

Sections 1.5, 1.6, and 1.7 on statistical physics, particle distributions, and transport processes emphasize topics which are fundamental to solid-state electronics and energy conversion. The second law of thermodynamics, equipartition of energy, and Maxwell distribution are obtained as natural consequences of energy conservation, and the law of large numbers, which states that the most likely states are observed. Fermi and Bose–Einstein distributions are derived with the introduction of elementary particle statistics. Transport effects such as electrical and thermal conduction, viscosity, diffusion, the Hall effect, and the various laws that relate these coefficients to one another are obtained as consequences of collision redistribution of energy, mass, and momentum.

1.1 BASICS IN PROBABILITY AND STATISTICS

The probability of an event is the likelihood of its happening, either from direct knowledge or from the recorded relative frequency of events of the kind in the course of experience.

For instance, the probability of any particular face turning up in throwing a die is 1/6, from our direct knowledge. The probability of rain at a given location in a given day of the year is calculated from accumulated data over the years. Many of the mathematical laws governing probability can be obtained intuitively. However, we can sharpen our concept on probability by associating it with a measure on a point set.

1.1.1 Probability as a Measure* on Elementary Events

Mathematically, probability can be defined as follows. Let x denote elements in a set W, and $\mu(x)$ a measure defined on each element x with the following properties:

$$\mu(x) \geqslant 0, \qquad x \in W \tag{1}$$

$$\underset{W}{\mathrm{S}}\,\mu(x) = 1 \tag{2}$$

*Many branches of mathematics are inspired by physical events. A well-known example is Newton's calculus, which was inspired or motivated by his study of kinetics. It is probably not far from the truth to say that measure theory is inspired by the study of probability. The correspondence between the two fields is so natural and obvious that previous knowledge of measure theory is not required for an understanding of the subsequent material. The reader may simply regard the term "measure" as synonymous with "assigned numerical value."

The symbol S represents either sum or integration, depending on whether x is discrete or continuous, and the symbol W underneath means that S is over the set W.

An event A is represented as a subset of W, and its probability $p(A)$ is defined as

$$p(A) = \underset{A}{\mathrm{S}}\,\mu(x) \tag{3}$$

Definition. Let $\{A_i\}$ denote the subsets A_i, $i = 1, 2, \ldots$. $\{A_i\}$ is said to be a *partition of* W if

$$A_i \cap A_j = 0, \qquad i \neq j \tag{4}$$

$$\cup A_i = W \tag{5}$$

In (4), 0 denotes a null set. It follows from (2) and (3) that

$$\sum_i p(A_i) = \underset{\underset{i}{\cup A_i}}{\mathrm{S}}\,\mu(x) = 1 \tag{6}$$

The reason that we equate probability with measure on a point set is as follows: For an event to occur, there are many, many possible ways. Each possible way is an element in W. Similarly, each possible way for it not to occur is also an element in W. A partition in W describes events that cannot occur jointly, but one of which must occur. For instance, in throwing a die, one and no more than one of the six faces must turn up.

The following theorem is useful in studying joint events.

THEOREM. If $\{A_i\}$ and $\{B_j\}$ are partitions of W, and

$$C_{ij} = A_i \cap B_j \tag{7}$$

then $\{C_{ij}\}$ is a partition of W.

The theorem is easily proved by showing that (4) and (5) are satisfied:

$$C_{ij} \cap C_{kl} = A_i \cap B_j \cap A_k \cap B_l$$

$$= (A_i \cap A_k) \cap (B_j \cap B_l)$$

If $(i, j) \neq (k, l)$, either $i \neq k$ or $j \neq l$. At least one of the intersection sets is a null set. Therefore, $C_{ij} \cap C_{kl}$ is a null set. Furthermore,

$$\bigcup_{ij} C_{ij} = \bigcup_j \bigcup_i (A_i \cap B_j) = \bigcup_j \left(\bigcup_i A_i \right) \cap B_j$$

$$= \bigcup_j B_j = W$$

1.1.2 Joint Probability, Conditional Probability, Independent Events

The correlation between different events is described by the notions of joint probability and conditional probability. They are defined as follows.

Joint Probability. Let A and B denote subsets of W. The joint probability $p(A, B)$ is defined as

$$p(A, B) = \underset{A \cap B}{\mathrm{S}}\,\mu(x) \tag{8}$$

It is the probability that both events A and B occur.

Conditional Probability. The knowledge that A has occurred is denoted by $|A$. It represents a reassignment of measure to elements x of W.

$$\mu(x|A) = 0 \qquad \text{if } x \notin A$$

$$\mu(x|A) = \frac{\mu(x)}{p(A)} \qquad \text{if } x \in A \tag{9}$$

Equation (9) means that the measure assigned to each $x \notin A$ is zero, and that assigned to each element $x \in A$ is multiplied uniformly by a factor $1/p(A)$ to satisfy

$$\underset{W}{\mathrm{S}}\,\mu(x|A) = \frac{\underset{A}{\mathrm{S}}\,\mu(x)}{p(A)} = 1$$

From (3), the conditional probability $p(B|A)$ becomes

$$p(B|A) = \underset{B}{\mathrm{S}}\,\mu(x|A)$$

$$= \underset{A\cap B}{\mathrm{S}} \frac{\mu(x)}{p(A)} = \frac{p(A,B)}{p(A)} \tag{10}$$

Equation (10) can be rewritten as

$$p(A)p(B|A) = p(A,B) = p(B)p(A|B) \tag{11}$$

It is known as *Bayes' theorem*.

Independent Events. The event A is *independent* of B if

$$p(A|B) = p(A) \tag{12}$$

Multiplying (9) by $p(B)$ gives

$$p(A,B) = p(A)p(B) \tag{13}$$

Equation (13) shows that (1) if A is independent of B, then B is independent of A; and (2) the joint probability $p(A,B)$ is then the product of the individual probabilities $p(A)p(B)$.

1.1.3 Discrete-Valued Random Variables

A discrete-valued random variable u is defined as follows. Let $\{A_i\}$ denote a partition of W. Each A_i is assigned a value u_i, and the probability that $u = u_i$ is

$$p(u_i) = p(A_i) = \underset{A_i}{\mathrm{S}}\,\mu(x) \tag{14}$$

It follows from (6) that

$$\sum_i p(u_i) = 1 \tag{15}$$

In a system with more than one random variable, say $u, v, w, \ldots$, for each variable there is a partition of W: $\{A_i\}$, $\{B_j\}$, $\{C_k\}, \ldots$, and $p(u_i), p(v_j), p(w_k), \ldots$, are correspondingly defined. The symbols $p(u_i), p(v_j)$, and so on, do not represent the same function with different independent variables. They represent the probabilities that $u = u_i$, $v = v_j$, and so on, and generally are different functions for different independent variables.

The joint probability that $u = u_i$ and $v = v_j$ is given by

$$p(u_i, v_j) = \underset{A_i\cap B_j}{\mathrm{S}}\,\mu(x) \tag{16}$$

Parallel to (9)–(11) we have similar expressions for conditional measure and conditional probability:

$$p(v_j|u_i) = \underset{B_j}{\mathrm{S}}\,\mu(x|A_i) = \frac{p(u_i, v_j)}{p(u_i)} \tag{17}$$

$$p(u_i)p(v_j|u_i) = p(u_i, v_j) = p(v_j)p(u_i|v_j) \tag{18}$$

The following relations are readily derived:

$$\sum_j p(u_i, v_j) = p(u_i) \tag{19}$$

$$\sum_i p(u_i, v_j) = p(v_j) \tag{20}$$

$$\sum_j p(v_j|u_i) = \sum_i p(u_i|v_j) = 1 \tag{21}$$

Quite often we ignore the subscripts and write the probabilities as $p(u)$, $p(u,v)$, and $p(u|v)$, respectively. The meanings remain the same.

If u and v are independent,

$$p(u|v) = p(u) \qquad \text{and} \qquad p(u,v) = p(u)p(v) \tag{22}$$

1.1.4 Continuous Random Variables

In some applications the possible value of a random variable u is continuous. The probability for u to be equal exactly to a given value u_1 is zero. However, given any tolerance interval $\pm\frac{1}{2}\delta u$, the probability of u being within the interval $u_1 - \delta u/2 \leq u \leq u_1 + \delta u/2$ is proportional to δu:

$$p\left(|u - u_1| < \frac{\delta u}{2}\right) = f(u_1)\,\delta u \tag{23}$$

The function f is known as the probability density or *density function*.

A more rigorous approach is as follows. The random variable u is defined by a measurable function $u(x)$ on W. Let $L(u_1)$ denote a subset in W such that

$$x \in L(u_1) \qquad \text{iff } u(x) \leq u_1 \tag{24}$$

The distribution function $F(u_1)$ is defined as

$$F(u_1) = \underset{L(u_1)}{\mathrm{S}}\ \mu(x) \tag{25}$$

From (3), $F(u_1)$ is recognized as the probability of $u \leq u_1$ and is an increasing function of u_1. $F(-\infty) = 0$ and $F(\infty) = 1$.

For continuous $F(u_1)$, the *density function* $f(u_1)$ is

$$f(u_1) = \frac{dF(u_1)}{du_1} \tag{26}$$

From (25) and (26), $f(u_1)$ can be written as

$$f(u_1) = \lim_{\Delta \to 0} \frac{1}{\Delta} \underset{D_\Delta}{\mathrm{S}}\ \mu(x) \tag{27}$$

where D_Δ represents the subset of x in W with $u(x)$ in the range $u_1 - \Delta/2 \leq u \leq u_1 + \Delta/2$. The equivalence of (23) and (26) is obvious.

For a multidimensional or vector-valued u, (27) can be generalized by considering Δ as a volume element about u_1, and D_Δ as the subset defined by

$$x \in D_\Delta \qquad \text{iff } u(x) \in \Delta \tag{28}$$

In the following, we shall use $F(u)$ and $f(u)$ to denote the distribution function and density function, respectively, of a random variable u.

A multiple number of random variables $u, v, \ldots$, can be considered as components of a vector $\mathbf{u}$. Using (27) instead of (8), the same steps leading to (11) give

$$f(u,v) = f(v)f(u|v) = f(u)f(v|u) \tag{29}$$

If u and v are independent, then

$$f(u,v) = f(u)f(v) \tag{30}$$

Similar to (15) and (21), we have

$$\int_{-\infty}^{\infty} f(u)\,du = F(\infty) - F(-\infty) = 1 \tag{31}$$

$$\int_{-\infty}^{\infty} f(v|u)\,dv = 1 \tag{32}$$

1.1.5 Average Value, Variance, Standard Deviation

The random variable u and its measurable functions are functions on W. For any given measurable function $v(x)$ on W, its average value $\langle v\rangle$ is defined as

$$\langle v\rangle = \underset{W}{\mathrm{S}}\, v(x)\mu(x) \tag{33}$$

The defining equation (31) has the following implications:

1. *Averaging is a linear process.* If $v(x)=v_1(x)+v_2(x)$, (33) gives

$$\langle v\rangle = \langle(v_1+v_2)\rangle = \langle v_1\rangle + \langle v_2\rangle \tag{34}$$

2. Let $g(u)$ denote a measurable function of the random variable u. It is then a function $g(u(x))$ of x. If u is discrete valued, (33) gives

$$\begin{aligned}\langle g(u)\rangle &= \underset{W}{\mathrm{S}}\, g(u(x))\mu(x)\\ &= \sum_i g(u_i)\underset{A_i}{\mathrm{S}}\,\mu(x) = \sum_i p(u_i)g(u_i)\end{aligned} \tag{35}$$

If u is continuous valued, we approximate $g(u)$ with a step function and then let the number of steps approach infinity, The end result is

$$\langle g(u)\rangle = \int_{-\infty}^{\infty} f(u)g(u)\,du \tag{36}$$

Equations (35) and (36) are also applicable when u is vector valued. The entities $f(u)$ and du are then the joint density of u's components and the volume element, respectively.

3. If u and v are independent, then (30) and the vector version of (36) give

$$\langle g(u)\cdot h(v)\rangle = \langle g(u)\rangle\cdot\langle h(v)\rangle \tag{37}$$

4. The *variance* of a random variable u is denoted σ^2 and is defined as the mean square deviation from the mean:

$$\sigma^2 = \langle(u-\langle u\rangle)^2\rangle = \langle(u^2-2u\langle u\rangle+\langle u\rangle^2)\rangle = \langle u^2\rangle - \langle u\rangle^2 \tag{38}$$

The constant σ is called the *standard deviation*.

5. Let u_i, $i=1,2,\ldots,N$, be N mutually independent events. Let u_t denote the sum

$$u_t = \sum_{i=1}^{i=N} u_i$$

Then

$$\begin{aligned}\langle u_t\rangle &= \sum_{i=1}^{i=N}\langle u_i\rangle\\ u_t-\langle u_t\rangle &= \sum_i (u_i-\langle u_i\rangle)\end{aligned} \tag{39}$$

$$\begin{aligned}\sigma_t^2 &= \left\langle \sum_i (u_i-\langle u_i\rangle)\sum_j (u_j-\langle u_j\rangle)\right\rangle\\ &= \sum_i \left\langle (u_i-\langle u_i\rangle)^2\right\rangle + \sum\sum_{i\neq j}\left\langle (u_i-\langle u_i\rangle)(u_j-\langle u_j\rangle)\right\rangle\end{aligned} \tag{40}$$

Since u_i and u_j, $i\neq j$, are independent,

$$\langle(u_i-\langle u_i\rangle)(u_j-\langle u_j\rangle)\rangle = \langle(u_i-\langle u_i\rangle)\rangle\langle(u_j-\langle v_j\rangle)\rangle = 0$$

and (40) becomes

$$\sigma_t^2 = \sum_i \sigma_i^2 \tag{41}$$

Equations (39) and (41) taken together has a very significant implication: The sum of N independent random variables has an average value proportional to N, but its standard deviation is proportional to $\sqrt{N}$. As N approaches infinity, the ratio of its standard deviation to its average value converges as $1/\sqrt{N}$. Thus the sum is more and more predictable as N increases. This is the *law of large numbers*. Its more refined form is the normal distribution, which is discussed in the next section.

6. The covariance coefficient between two random variables u and v is defined as

$$C_{uv} = \langle (u - \langle u \rangle)(v - \langle v \rangle) \rangle \tag{42}$$

Let σ_u and σ_v denote the standard deviations of u and v, respectively. Then

$$\left(\frac{|u - \langle u \rangle|}{\sigma_u} - \frac{|v - \langle v \rangle|}{\sigma_v} \right)^2 \geqslant 0$$

Expanding and averaging the expression above gives

$$2 - \frac{2\langle |u - \langle u \rangle| \, |v - \langle v \rangle| \rangle}{\sigma_u \sigma_v} \geqslant 0 \tag{43}$$

Since

$$\langle |u - \langle u \rangle| \, |v - \langle v \rangle| \geqslant |C_{uv}|$$

(43) gives

$$|C_{uv}| \leqslant \sigma_u \sigma_v \tag{44}$$

The equality sign of (44) holds if

$$\frac{u - \langle u \rangle}{\sigma_u} \pm \frac{v - \langle v \rangle}{\sigma_v} = 0 \tag{45}$$

That is, u and v are linearly dependent. The matrix

$$\mathbf{C} = \begin{pmatrix} \sigma_u^2 & C_{uv} \\ C_{uv} & \sigma_v^2 \end{pmatrix} \tag{46}$$

is called the *covariance matrix*. It is positive definite if the two random variables u and v are not linearly dependent.

The definition of covariance matrix can be extended to m-variables $\mathbf{u} = (u_1, u_2, \ldots, u_m)$: C is a $m \times m$ matrix with

$$C_{ij} = \langle (u_i - \langle u_i \rangle)(u_j - \langle u_j \rangle) \rangle \tag{47}$$

The matrix $\mathbf{C}$ is positive definite if no such linear dependence exist:

$$a_0 + \sum_{i=1}^{i=m} a_i u_i = 0 \tag{48}$$

EXAMPLE 1. In analog-to-digital conversion, a continuous variable u is represented as nq if

$$|u - nq| \leqslant \frac{q}{2} \tag{49}$$

where q is the separation between quantized levels. Assume that the density function $f(u)$ is constant, and show that

$$\langle e \rangle = 0$$

$$\langle e^2 \rangle = \frac{q^2}{12} \tag{50}$$

where $e = u - nq$.

Solution. Since (49) is a condition of u being represented as nq, the conditional $f(u|n)$ is

$$f(u|n) = \begin{cases} \frac{1}{q} & \text{if (49) is satisfied} \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

$$\langle e \rangle = \int_{-q/2}^{q/2} \frac{e}{q} de = 0$$

$$\langle e^2 \rangle = \int_{-q/2}^{q/2} \frac{e^2}{q} de = \left[\frac{e^3}{3q} \right]_{-q/2}^{q/2} = \frac{q^2}{12} \tag{52}$$

1.1.6 Conditional Average

If $u(x)$ in (33) is replaced by $u(x|A)$, that is,

$$\langle v \rangle = \underset{W}{\mathrm{S}}\, v(x) u(x|A) \tag{53}$$

where $W \supseteq A$, $\langle v \rangle$ is then the conditional average. The properties 1 through 6 of the preceding section remain valid. Equations (35) and (36) are replaced by

$$\langle g(u) \rangle = \sum_i p(u_i|A) g(u_i) \tag{54}$$

for a discrete random variable u, and

$$\langle g(u) \rangle = \int_{-\infty}^{\infty} f(u|A) g(u)\, du \tag{55}$$

for a continuous-valued u.

1.2 STATISTICAL MOMENTS AND DISTRIBUTIONS

In the present section we define statistical moments, characteristic and moment-generating functions, and derive a number of well-known distributions. In a multiple-random-variable system, any average value of the form

$$\langle u^{n_1} v^{n_2} w^{n_3} \rangle$$

is called an *nth moment*, where $n_1, n_2, \ldots,$ are nonnegative integers, and $n = n_1 + n_2 + \cdots$. The average value of a random variable is its first moment. The variance and covariance are second moments.

1.2.1 Moment-Generating Function, Characteristic Function

Let $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ denote a random vector with m-components. Its moment-generating function is defined to be

$$M(\mathbf{s}) = \int_{-\infty}^{\infty} f(\mathbf{u}) \exp(\mathbf{s} \cdot \mathbf{u})\, d\mathbf{u} \tag{1}$$

where the integral sign and $d\mathbf{u}$ are understood to be m-dimensional. $M(\mathbf{s})$ is recognized as the Laplace transform of $f(\mathbf{u})$ with $-\mathbf{s}$ as the transform variable. It is also the average value of $\exp(\mathbf{s} \cdot \mathbf{u})$. Let the subscripts $a, b, c, \ldots,$ represent integers from 1 to m which may or may not be repeated. Then

$$\frac{\partial\partial\cdots\partial\exp(\mathbf{s}\cdot\mathbf{u})}{\partial s_a\partial s_b\partial s_c} = (u_a u_b u_c\cdots)\exp(\mathbf{s}\cdot\mathbf{u})$$

and

$$\left[\frac{\partial\partial\cdots\partial M(\mathbf{s})}{\partial s_a\partial s_b\partial s_c\cdots}\right]_{\mathbf{s}=0} = \langle u_a u_b u_c\cdots\rangle \tag{2}$$

Equation (2) shows that the statistical moments are the corresponding partial derivatives of the moment-generating function evaluated at the origin.

For purely imaginary values of s, $|\exp(\mathbf{s}\cdot\mathbf{u})| = 1$ and (1) gives

$$|M(\mathbf{s})| \leq \int f(\mathbf{u})\,d\mathbf{u} = 1 \tag{3}$$

Therefore, the integral of (1) is convergent in the neighborhood of

$$\mathbf{s} = j\omega \tag{4}$$

The *characteristic function* $C(\omega)$ is obtained by replacing s in (1) with $j\omega$: It is the Fourier transform of the density function

$$C(\omega) = \int_{-\infty}^{\infty} f(\mathbf{u})\exp(j\omega\cdot\mathbf{u})\,d\mathbf{u} \tag{5}$$

Sometimes the characteristic function is obtained first, and $f(\mathbf{u})$ is calculated from it by using the inverse Fourier transform:

$$f(\mathbf{u}) = \frac{1}{(2\pi)^m}\int_{-\infty}^{\infty} C(\omega)\exp(-j\omega\cdot\mathbf{u})\,d\omega \tag{6}$$

EXAMPLE 1. A random variable u has the following density functions

$$f(u) = \frac{a}{2}e^{-a|u|}$$

Determine its statistical moments and characteristic function.

Solution. The Laplace transform of $f(u)$ is

$$M(s) = \frac{a}{2}\left(\frac{1}{a+s} + \frac{1}{a-s}\right) = \frac{a^2}{a^2-s^2}$$

$$= \frac{1}{1-s^2/a^2} = 1 + \frac{s^2}{a^2} + \frac{s^4}{a^4} + \cdots$$

There are no odd moments; the even moments are given by

$$M_n = \left.\frac{d^nM(s)}{ds^n}\right]_{s=0} = \frac{n!}{a^n}$$

The characteristic function is

$$C(\omega) = M(j\omega) = \frac{a^2}{\omega^2 + a^2}$$

1.2.2 Multivariate Normal Distribution

The *multivariate normal distribution* (MND) of m random variables $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ can be expressed in terms of its density function:

$$f(\mathbf{u}) = \frac{1}{(2\pi)^{m/2}\Delta^{1/2}}\exp\left[-\tfrac{1}{2}(\mathbf{u}-\bar{\mathbf{u}})C^{-1}(\mathbf{u}-\bar{\mathbf{u}})^T\right] \tag{7}$$