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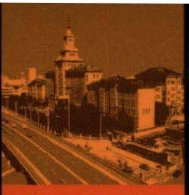
Approximation and Optimization of Discrete and Differential Inclusions



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离散与微分包含的逼近和优化

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Elimhan N. Mahmudov
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Dedication

This book is dedicated to the memory of my doctoral thesis advisor B.N. Pshenichnyi, Academician of the Academy of Sciences of Ukraine, from whom I learned the theory of extremal problems, numerical methods in optimal systems theory, optimal control, differential game theory, and much more.

This book is also dedicated to memory of my lovely son Elshad (1977–2000), who was a selfless manager for the American petrol company Exxon.

Besides this book is dedicated to the memory of Khojali massacre's martyrs (25-26.02.1992).

Preface

Give me a place to stand on, and I will move the Earth.

Archimedes

Mathematics is the language with which God has written the universe.

Galileo Galilei

As a well spent day brings happy sleep, so life well used brings happy death

Leonardo da Vinci

Scientists investigate that which already is;engineers create that which has never been

Albert Einstein

The primary goals of this book are to present the basic concepts and principles of mathematical programming in terms of set-valued analysis (Chapters 2 and 3) and on the basis of the method of approximation, to develop a comprehensive optimality theory of problems described by ordinary and partial differential inclusions (DFI) (Chapters 4–6). This book consists of six chapters divided into sections and subsections, and contains many results that have not been published in the monographic literature.

In Chapter 1, convex sets and convex functions are studied in the setting of n-dimensional Euclidean space. However, the reader familiar with functional analysis can generalize the main results to the case of infinite-dimensional functional spaces. In spite of the fact that the stated notions and results are known, they play a decisive role for obtaining the main results in the next chapters of the book. The key issues of convex analysis in finite-dimensional spaces have been addressed in the books *Convex Analysis* by Rockafellar and *Convex Analysis and Extremum Problems* by B.N. Pshenichnyi. The identifications of convex functions and their epigraphs make it easy to pass back and forth between the geometric and analytical approaches. It is shown that convex sets and functions form classes of objects preserved under numerous operations of combination; pointwise addition, pointwise supremum, and infimal convolution of convex functions are convex.

In Chapter 2, the apparatus of locally adjoint mappings (LAM) (which is new) is studied in the light of convex analysis. It is the fundamental concept in what follows, and it is used to obtain the optimality conditions for the problems posed in this book. We give the calculus of LAM on different multivalued mappings, such as the sum, composition, and inverse. We introduce the adjoint (not locally adjoint) mapping, using the recession cone, and the connection between the adjoint and LAM is established. Based on the adjoint mapping, the duality theorems for convex set-valued mappings are proved.

Chapter 3 is devoted to applications of these basic tools to the study of mathematical programming with possibly nonsmooth data. Starting with problems of mathematical programming under functional and geometric constraints, we then consider various problems of constrained optimization, minimax problems and equilibrium constraints, infimal convolution of convex functions, duality in convex programming, and duality relations. In order to formulate a necessary condition for the existence of an extremum, of course, some special condition by function taking part in the given problem is required. In particular, in some neighborhood of a point minimizing our objective function, we deal with the comparatively easily computable functions. As is known, a smooth function admits a linear approximation. On the other hand, a convex function can be approached by positively homogeneous functions. However, a nonsmooth and nonconvex function cannot be approximated in a neighborhood of a point by positively homogeneous functions. Precisely this class of functions is required for introducing the concept of convex upper approximations (CUAs). The key tools of our analysis are based on the extremal principle and its modifications together with the LAM calculus.

Chapters 4 and 5 deal mostly with optimal control problems of the Bolza type described by ordinary differential, high-order differential, delay-differential, and neutral functional-differential inclusions. The development and applications of the LAM are demonstrated in these problems with ordinary discrete and differential inclusions. In particular, for polyhedral DFI, under the corresponding condition for generality of position, the theorem of the number of switchings is proved. The corresponding results are obtained for linear optimal control problems in linear manifolds. For a nonautonomous polyhedral DFI, a special condition for generality of position is formulated. Moreover, for problems described by ordinary nonconvex DFI under the specially formulated monotonicity and t_1-transversality conditions, sufficient conditions for optimality are proved.

In Chapter 6, we continue the study of optimal control problems governed by discrete and differential inclusions with distributed parameters, which during the past 15–20 years has been a basic source of inspiration for analysis and applications. Using LAM and the discrete-approximation method in Hamiltonian and Euler–Lagrange forms, we derive necessary and sufficient optimality conditions for various boundary values (Dirichlet, Neumann, Cauchy) problems for first-order, elliptic, parabolic, and hyperbolic types of discrete and partial DFI. One of the most characteristic features of such approaches with partial DFI is peculiar to the presence of equivalents to the LAM. Such problems have essential specific features in comparison with the ordinary differential model considered in the second part of the book. For every concrete problem with partial DFI, we establish rather interesting equivalence results that shed new light on both qualitative and quantitative relationships between continuous and discrete approximation problems.

In Chapter 5 and the second part of Chapter 6, we construct the dual problem of convex problems for ordinary and partial differential inclusions of hyperbolic, parabolic, and elliptic types. We study separately the duality problems with first-order partial differential inclusions. As is known, duality problems have always been at the center of convex optimality theory and its applications. In this book, we formulate duality results and search for the conditions under which primary and dual problems are connected by such duality relations. For duality constructions of convex problems, we use the duality theorems concerning infimal convolution and the addition of convex functions.

Thus, we can list the major features of our book that make it unique:

- The introduction of a new concept of LAM and its calculus.
- The connection between LAM and adjoint (not locally) mappings defined in terms of the recession cone.
- Duality theorems for convex multivalued mappings established in terms of a Hamiltonian function.
- The basic results of mathematical programming in terms of Hamiltonian functions.
- Under a suitable condition for generality of position, the theorem of the finiteness of switching numbers for optimal control of polyhedral differential inclusions.
- Under a special t_1-transversality condition, new sufficient conditions for optimality in terms of extended Euler–Lagrange inclusions for Bolza-type problems with ordinary differential inclusions and state constraints.
- A new class of optimal control problems for higher-order differential inclusions.
- Duality relations in mathematical problems with equilibrium constraints via recession cones. Major features using the method of discrete approximation.
- Optimization of first-order discrete and partial differential inclusions. Note that one of the characteristic features of optimization of Cauchy for first-order discrete inclusions is the intrinsic presence of the infinite dimensional self-adjoint Hilbert space l_2.
- Optimization of Darboux-type partial differential inclusions and duality.
- Optimization of elliptic, hyperbolic, and parabolic types of discrete and partial differential inclusions and duality.
- Optimization of partial differential inclusions with a second-order elliptic operator.
- Equivalence results that facilitate making a bridge between discrete and corresponding discrete-approximation problems.

Throughout this book, a proof is marked with an empty square □ and its end is marked with a Halmos box, ■. Since many problems in engineering reduce to such problems, the book will be of interest to mathematicians and nonmathematician specialists whose study involves the use of ordinary and partial differential equations (inclusions) and approximation methods and its applications, as well as to undergraduate, graduate, and postgraduate students at universities and technical colleges. In other words, the book is intended for a broad audience—students of universities and colleges with comprehensive mathematical programs, engineers, economists, and mathematicians involved in the solution of extremal problems.

Basic material has also been incorporated into many lectures given by the author at various international conferences in London, UK; Zurich, Switzerland; and Leipzig, Germany, and the Banach international mathematical center in Warsaw, Poland, during recent years.

This book includes an index of symbols and notation. Using this index, the reader can easily find the page where some notion or notation is introduced. Our notation and terminology are generally consistent with those used by Rockafellar, Mordukhovich, and Pshenichnyi in their writings. For the reader's convenience, an introduction in each chapter of the book describes the contents and commentaries, and outlines a selection of material that would be appropriate for the subject. This book is also accompanied by an abundant bibliography. Parts of this book have been used by me in teaching graduate and postgraduate students on Convex Analysis, Optimal Control Theory, and Nonlinear Analysis and Its Applications at Azerbaijan State University and Istanbul Technical University.

Prof. Elimhan N. Mahmudov
Baku and Istanbul
August 2011

Acknowledgments

My first thanks go to Boris S. Mordukhovich, Professor in the Department of Mathematics at Wayne State University, who inspired me during the work on this book. I am particularly indebted to Alex Rubinov, V.F. Demyanov, and A.V. Kryazhimskii at all stages of this book for recognizing and supporting my first efforts in the field of locally adjoint mappings and optimization. I am indebted to the Institute of Cybernetics, headed by T.A. Aliev, Academician of the National Academy of Sciences of Azerbaijan. I would like to thank my genial colleagues C. Kahraman, N. Serarslan, and S. Polat of Istanbul Technical University for their support during the preparation of this book. I am very grateful to the supportive editors at Elsevier for their help during the preparation and publishing of this book; the timely publication of this book is a direct consequence of the help of Lisa Tickner, Publishing Director at Elsevier. Especially valuable help was provided by my doctoral students Özkan Değer, Murat Ünal, and Kutay Tinç. And I thank my pencil sharpener, my younger daughter Elmira, for her interest in my book.

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About the Author

Elimhan N. Mahmudov was born in Kosali, Karayazi, and after graduating the secondary school with a gold medal for being the most successful student, he attended Azerbaijan State University as part of the Mechanical–Mathematical Department. From 1973 to 1977, he was a PhD student at the Cybernetics Institute of Academy of Sciences of Ukraine advised by Prof. B.N. Pshenichnyi—Academician, one of the most prominent authorities in the fields of theory of extremal problems, numerical methods, and game theory. E.N. Mahmudov defended his thesis in Ukraine in 1980. He then worked as a Ph.D. and senior scientific associate at the Cybernetics Institute of Academy of Sciences of Azerbaijan until 1995. He also taught part time at the Azerbaijan State University and was a permanent member of the Physical–Mathematical Doctoral Sciences Committee in Azerbaijan. On the recommendation of the Physical–Mathematical Doctoral Committee of the Taras Shevchenko's Kiev State University, in 1992 he became a Doctor of Physical–Mathematical Sciences, and he has earned both a Ph.D. and a Doctorate of Sciences from Moscow. In 1992, he received a Grant Support for Mathematics by National Science Foundation in Washington, DC, and became a member of the American Mathematical Society.

Prof. Mahmudov has devoted numerous research papers to Convex Analysis, Approximation Theory, Optimal Control Theory, Dual Problems, and Mathematical Economy, which have been published in high-level Science Citation Index (SCI) journals in the former Soviet Union, the United States, and Europe. His textbooks include *Mathematical Analysis and Its Applications*, published in 2002 by Papatya, Istanbul, and *Single Variable Differential and Integral Calculus* (in press). He has lectured on the problems of Optimal Control at International Conferences in England, France, Switzerland, Germany, Poland, Russia, and Turkey, and at the World Mathematical Banach Center in Warsaw, Poland. He is an Editor of the journal *Advances in Pure Mathematics* (APM).

Prof. Mahmudov currently teaches Nonlinear Programming, Game Theory, and applications in Economics, Nonlinear Optimization, and Numerical Analysis at Istanbul Technical University. He enjoys playing chess, playing Tar and Saz, Azerbaijan folk instruments, and traveling and is an avid painter.

Contents

1 Convex Sets and Functions

1.1 Introduction

Convexity is an attractive subject to study, for many reasons; it draws upon geometry, analysis, linear algebra, and topology, and it has a role to play in such topics as classical optimal control theory, game theory, linear programming, and convex programming. Convex sets and convex functions are studied in this chapter in the setting of n-dimensional Euclidean space $\mathbb{R}^n$. (However, if you are familiar with functional analysis, you will be able to generalize the main results to the case of infinite dimensional functional spaces.) These results play a decisive role in obtaining the main results in the next chapters of this book. In the field of convex analysis, you can consult Rockafellar [228] and Pshenichnyi [226] for related and additional material (most of the familiar results on convex analysis presented in this chapter are taken from Pshenichnyi [226]). The basic idea in convexity is that a convex function on $\mathbb{R}^n$ can be identified with a convex subset of $\mathbb{R}^{n+1}$, which is called the epigraph of the given function. This identification makes it easy to move back and forth between geometrical and analytical approaches. It is shown that pointwise addition of functions, pointwise supremum, and infimal convolution of convex functions are convex, in fact, convex sets and functions are classes of objects that are preserved under numerous operations of combination. A function is closed if its epigraph is closed. The latter is equivalent to lower semicontinuity of functions. This leads to the notion of the closure operation for proper convex functions, which corresponds to the closure operation for epigraphs.

In Section 1.2, we study some topological properties of sets and their convex hull, and consider how a convex set can be characterized by both Minkowski's method and support functions. The role of dimensionality in the generation of convex hulls is explored in Carathéodory's theorem (Theorem 1.1). It is interesting that Theorem 1.2 (Gauss–Lucas) says that the roots of the derivative of a nonconstant complex polynomial belong to the convex hull of the set of roots of the polynomial itself. The foundations for extremal theory are laid in the Separation Theorems 1.5–1.7.

In Section 1.3, we discuss the convex cone, which is one of the important concepts in convex analysis and extremal theory. The investigation of its properties is connected with the calculation of the dual cone.

The cones $K_1, \dots, K_m$ are called separable if there exist not all zero vectors $x_i^* \in K_i^*$, such that $x_1^* + \cdots + x_m^* = 0$. By Theorem 1.12, if $K = K_1 \cap \cdots \cap K_m$, then either $K^* = K_1^* + \cdots + K_m^*$ or the cones $K_1, \dots, K_m$ are separable. By Lemma 1.17,

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$(K_1 \cap \cdots \cap K_m)^* = \overline{K_1^* + \cdots + K_m^*}$. But since for polyhedral cones, $K_1^* + \cdots + K_m^*$ is also a polyhedral cone, this sum of cones is closed and the bar above it can be removed. One of the remarkable properties of a polyhedral set is that it can be represented as a sum of polytope (polyhedron) and polyhedral cone. Conversely, the sum of any polytope and polyhedral cone is a polyhedral set (Theorem 1.14). The recession cone of a nonempty convex set M, i.e., the set of vectors $\bar{x}$ such that $M + \bar{x} \subset M$ is denoted by $0^+ M$ and for a bounded set $0^+ M = \{0\}$.

In Section 1.4, we develop the main properties of convex functions. Recall that by Definition 1.20 a function f is said to be proper, if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x. A function that is not proper is improper. It follows from Lemma 1.24 that dom f is convex, even if f is an improper function. Besides, an improper convex function may have a finite values only at points of the relative boundary of dom f. The sum of proper convex functions f_i, $i = 1, \ldots, m$ with nonnegative coefficients is convex (Lemma 1.27).

It is known that the indicator function $\delta_M(\cdot)$ of M is useful as a correspondence between convex sets and convex functions. Then note that the sum of proper convex functions f_i, $i = 1, \ldots, m$ with nonnegative coefficients may not be a proper function (Lemma 1.27). For example, for the disjoint sets M_1 and M_2, the sum of indicator functions $\delta_{M_1} + \delta_{M_2}$ is identically $+\infty$.

We shall denote the gradient of f at x by $f'(x)$ and the Hessian matrix of f at x by $f''(x)$, whose (i,j)th element is $\partial^2 f / \partial x^i \partial x^j$. Then if f is twice differentiable, the convexity of f and the positive semidefiniteness of $f''(x)$ are equivalent. Of course, the latter is an important result not only in analysis but also in nonlinear programming.

Lemma 1.29 implies that properness of convex functions is not always preserved by infimal convolution $f_1 \oplus f_2$, which is commutative, associative, and convexity-preserving. Indeed, if f_1 and f_2 are linear functions not equal to each other, then their infimal convolution identically is $-\infty$.

The convex hull conv g of a nonconvex function g, defined as the greatest convex function majorized by g, is used in establishing the dual problem governed by polyhedral maps in Section 5.2. In Theorems 1.16 and 1.17, the continuity and Lipschitz properties of convex functions are shown. By Theorem 1.18, f, a proper convex function, is necessarily continuous on ri dom f. As is seen from this theorem, a convex function is continuous in dom f and may have a point of discontinuity only in its boundary. In order to characterize the case in which there is no such discontinuity, it is convenient to introduce the closure function concept (a function f is said to be a closure if its epigraph epi f is a closed set in $\mathbb{R}^{n+1}$). By Definition 1.26, the recession function is denoted by $f0^+$ and defined as epi $(f0^+) = 0^+$ (epi f). Obviously, if f is a proper convex function, then the recession function $f0^+$ of f is a positively homogeneous proper convex function.

Section 1.5 is devoted to the conjugate of a convex function, which is one of the basic concepts both of convex analysis and of duality theory. The definition of the conjugate of a function grows naturally out of the fact that the epigraph of a closed proper convex function on $\mathbb{R}^n$ is the intersection of the closed half-spaces in $\mathbb{R}^{n+1}$ that contain it. The function defined as $f^*(x^*) = \sup_x\{\langle x, x^* \rangle - f(x)\}$ is called the conjugate of f. It is closed and convex. It is useful to remember, in particular, that