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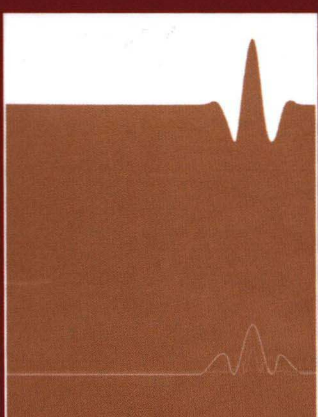
Sparse Image and Signal Processing



Wavelets and Related Geometric Multiscale Analysis

Second Edition

SPARSE IMAGE AND SIGNAL PROCESSING

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Second Edition

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Sparse Image and Signal Processing
Wavelets and Related Geometric Multiscale Analysis
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This thoroughly updated new edition presents state-of-the-art sparse and multiscale image and signal processing. It covers linear multiscale geometric transforms, such as wavelet, ridgelet, or curvelet transforms, and nonlinear multiscale transforms based on the median and mathematical morphology operators. Along with an up-to-the-minute description of required computation, it covers the latest results in inverse problem solving and regularization, sparse signal decomposition, blind source separation, inpainting, and compressed sensing. New chapters and sections cover multiscale geometric transforms for three-dimensional data (data cubes), data on the sphere (geolocated data), dictionary learning, and nonnegative matrix factorization.

The authors wed theory and practice in examining applications in areas such as astronomy, including recent results from the European Space Agency's Herschel mission, biology, fusion physics, cold dark matter simulation, medical MRI, digital media, and forensics. MATLAB® and IDL code, available online at www.SparseSignalRecipes.info, accompany these methods and all applications.

Jean-Luc Starck is Senior Scientist at the Institute of Research into the Fundamental Laws of the Universe, Commissariat à l'Énergie Atomique de Saclay, France. His research interests include cosmology, weak lensing data, and statistical methods such as wavelets and other sparse representations of data. He has published more than 200 papers in astrophysics, cosmology, signal processing, and applied mathematics, and is also the author of three books.

Fionn Murtagh has served in the Space Science Department of the European Space Agency for twelve years. He is a Fellow of both the International Association for Pattern Recognition and the British Computer Society, as well as an elected member of the Royal Irish Academy and of Academia Europaea. He is a member of the editorial boards of many journals and has been editor-in-chief of *The Computer Journal* for more than ten years.

Jalal M. Fadili has been a full professor at Institut Universitaire de France since October 2013. His research interests include signal and image processing, statistics, optimization theory, and low-complexity regularization. He is a member of the editorial boards of several journals.

List of Acronyms

1-D, 2-D, 3-D	one-, two-, three-dimensional
AAC	Advanced Audio Coding
ADMM	Alternating-Direction Method of Multipliers
AIC	Akaike Information Criterion
BCR	Block-Coordinate Relaxation
BIC	Bayesian Information Criterion
BP	Basis Pursuit
BPDN	Basis Pursuit DeNoising
BSS	Blind Source Separation
BSE	Back Scattered Electron
BT	Beamlet Transform
CCD	Charge Coupled Device
CeCILL	CEA CNRS INRIA Logiciel Libre
CIF	Common Intermediate Format
CMB	Cosmic Microwave Background
CTS	Curvelet Transform on the Sphere
CurveletG1, G2	first, second generation curvelet
CS	Compressed Sensing
CWT	Continuous Wavelet Transform
dB	Decibel
DCT	Discrete Cosine Transform
DCTG1, DCTG2	Discrete Curvelet Transform, first/second Generation
DR	Douglas-Rachford
DRT	Discrete Ridgelet Transform
DWT	Discrete Wavelet Transform
ECP	Equidistant Coordinate Partition
EEG	ElectroEncephaloGraphy
EFICA	Efficient Fast Independent Component Analysis
EM	Expectation Maximization
ERS	European Remote Sensing
ESO	European Southern Observatory

ESA	European Space Agency
FB	Forward-Backward
FCT	Fast Curvelet Transform
FDCT	Fast Discrete Curvelet Transform
FDR	False Discovery Rate
FFT	Fast Fourier Transform
FIR	Finite Impulse Response
FISTA	Fast Iterative Shrinkage-Thresholding Algorithm
FITS	Flexible Image Transport System
fMRI	functional Magnetic Resonance Imaging
FOCUSS	FOcal Underdetermined System Solver
FSS	Fast Slant Stack
FWER	Family-Wise Error Rate
FWHM	Full Width at Half Maximum
GCV	Generalized Cross-Validation
GFB	Generalized Forward-Backward
GLESP	Gauss-Legendre Sky Pixelization
GMCA	Generalized Morphological Component Analysis
GUI	Graphical User Interface
HALS	Hierarchical Alternating Least Squares
HEALPix	Hierarchical Equal Area isoLatitude Pixelization
HSD	Hybrid Steepest Descent
HTM	Hierarchical Triangular Mesh
ICF	Inertial Confinement Fusion
ICA	Independent Component Analysis
IDL	Interactive Data Language
IFFT	Inverse FFT
IHT	Iterative Hard Thresholding
iid	Independently and Identically Distributed
IRAS	Infrared Astronomical Satellite
IRLS	Iterative Reweighted Least-Squares
ISO	Infrared Space Observatory
IST	Iterative Soft Thresholding
ISTA	Iterative Soft Thresholding Algorithm
ITZ	Interfacial Transition Zone
IUWT	Isotropic Undecimated Wavelet (starlet) Transform
JADE	Joint Approximate Diagonalization of Eigen-matrices
JPEG	Joint Photographic Experts Group
KL	Kullback-Leibler
KL	Kurdyka-Łojasiewicz
LARS	Least Angle Regression
LAT	Large Area Telescope
lhs	lefthand side
LoG	Laplacian of Gaussian
LP	Linear Programming
LR-FCT	Low Redundancy Fast Curvelet Transform

lsc	lower semi-continuous
MAD	Median Absolute Deviation
MAP	Maximum a Posteriori
MCA	Morphological Component Analysis
MDL	Minimum Description Length
MGA	Multiscale Geometric Analysis
MI	Mutual Information
ML	Maximum Likelihood
MM	Majorization-Minimization
MMT	Multiscale Median Transform
MMV	Multiple Measurements Vector
MOLA	Mars Orbiter Laser Altimeter
MOM	Mean of Max
MP	Matching Pursuit
Mpc	Mega parsecs
MP3	MPEG-1 Audio Layer 3
MPEG	Moving Picture Experts Group
MPI	Message Passing Interface
MR	Magnetic Resonance
MRF	Markov Random Field
MSE	Mean Square Error
MS-VST	Multiscale Variance Stabilization Transform
NLA	Nonlinear Approximation
NASA	National Aeronautics and Space Administration
NMF	Nonnegative Matrix Factorization
NOAA	National Oceanic and Atmospheric Administration
NP	Non-Polynomial
NRMSE	Normalized Root Mean Square Error
OFRT	Orthonormal Finite Ridgelet Transform
OSCIR	Observatory Spectrometer and Camera for the Infrared
OWT	Orthogonal Wavelet Transform
PACS	Photodetector Array Camera and Spectrometer
PCA	Principal Components Analysis
PCTS	Pyramidal Curvelet Transform on the Sphere
PDE	Partial Differential Equation
PDF	Probability Density Function
PMT	Pyramidal Median Transform
PNG	Portable Network Graphics
POCS	Projections Onto Convex Sets
PPFT	Pseudo-Polar Fourier transform
PSF	Point Spread Funcation
PSNR	Peak Signal to Noise Ratio
PTF	Parseval Tight Frame
PWT	Partially decimated Wavelet Transform
PWTS	Pyramidal Wavelet Transform on the Sphere
QMF	Quadrature Mirror Filter

rhs	righthand side
RIC	Restricted Isometry Constant
RIP	Restricted Isometry Property
RNA	Relative Newton Algorithm
RTS	Ridgelet Transform on the Sphere
SAR	Synthetic Aperture Radar
SBT	Spherical Bessel Transform
SDR	Source-to-Distortion Ratio
SeaWiFS	Sea-viewing Wide Field-of-view Sensor
SEM	Scanning Electron Microscope
SFB	Spherical Fourier-Bessel
SNR	Signal-to-Noise Ratio
s.t.	subject to
SSRT	Slant Stack Radon Transform
STFT	Short-Time Fourier Transform
SURE	Stein Unbiased Risk Estimator
USFFT	Unequi-Spaced FFT
UWT	Undecimated Wavelet Transform
UWTS	Undecimated Wavelet Transform on the Sphere
VST	Variance Stabilizing Transform
WMAP	Wilkinson Microwave Anisotropy Probe
WT	Wavelet Transform
TV	Total Variation

Notation

Functions and signals

$f(t)$	Continuous-time function, $t \in \mathbb{R}$.
$f(\mathbf{t})$ or $f(t_1, \ldots, t_d)$	d-D continuous-time function, $\mathbf{t} \in \mathbb{R}^d$.
$f[k]$	Discrete-time signal, $k \in \mathbb{Z}$, or kth entry of a finite dimensional vector.
$f[\mathbf{k}]$ or $f[k, l, \ldots]$	d-D discrete-time signal, $\mathbf{k} \in \mathbb{Z}^d$.
$\bar{f}$	Time-reversed version of f as a function ($\bar{f}(t) = f(-t), \forall t \in \mathbb{R}$) or signal ($\bar{f}[k] = f[-k], \forall k \in \mathbb{Z}$).
$\hat{f}$	Fourier transform of f.
f^*	Complex conjugate of a function or signal.
$H(z)$	z-transform of a discrete filter h.
$\text{lhs} = O(\text{rhs})$	lhs is of order rhs; there exists a constant $C > 0$ such that $\text{lhs} \leq C\text{rhs}$.
$\text{lhs} \sim \text{rhs}$	lhs is equivalent to rhs; $\text{lhs} = O(\text{rhs})$ and $\text{rhs} = O(\text{lhs})$.
$\mathbf{1}_{\{\text{condition}\}}$	1 if condition is met, and zero otherwise.
$L_2(\Omega)$	Space of square-integrable functions on a continuous domain Ω.
$\ell_2(\Omega)$	Space of square-summable signals on a discrete domain Ω.
$\Gamma_0(\mathcal{H})$	Class of proper lower-semicontinuous convex functions from $\mathcal{H}$ to $\mathbb{R} \cup \{+\infty\}$.

Operators on signals or functions

$[\cdot]_{\downarrow 2}$	Down-sampling or decimation by a factor two.
$[\cdot]_{\downarrow 2^e}$	Down-sampling by a factor two that keeps even samples.
$[\cdot]_{\downarrow 2^o}$	Down-sampling by a factor two that keeps odd samples.
$\breve{\cdot}$ or $[\cdot]_{\uparrow 2}$	Up-sampling by a factor two (i.e. zero-insertion between each two samples).
$[\cdot]_{\uparrow 2^e}$	Even-sample zero-insertion.
$[\cdot]_{\uparrow 2^o}$	Odd-sample zero-insertion.

$[\cdot]_{\downarrow 2,2}$	Down-sampling or decimation by a factor two in each direction of a 2D image.
$*$	Continuous convolution.
$\star$	Discrete convolution.
$\bullet$	Composition (arbitrary).
∂F	Subdiffential of F.
∇F	Gradient of F
prox_F	Proximity operator of F.
$\mathrm{SoftThresh}_\lambda$	Soft-thresholding with threshold λ.
$\mathrm{HardThresh}_\lambda$	Hard-thresholding with threshold λ.

Sets

$\mathrm{dom}(F)$	Domain of function F.
$\mathrm{ri}(\mathcal{C})$	Relative interior of set $\mathcal{C}$.
$\iota_{\mathcal{C}}$	Indicator function of set $\mathcal{C}$.
$\mathrm{P}_{\mathcal{C}}$	Orthogonal projector on $\mathcal{C}$.

Matrices, linear operators, and norms

Bold capital symbols	Matrices or linear operators (e.g. $\mathbf{M}$).
$.^{\mathrm{T}}$	Transpose of a vector or a matrix.
$\mathbf{M}^*$	Adjoint of $\mathbf{M}$.
Gram matrix of $\mathbf{M}$	$\mathbf{M}^*\mathbf{M}$ or $\mathbf{M}^{\mathrm{T}}\mathbf{M}$.
$\mathbf{M}[i, j]$	Entry at ith row and jth column of a matrix $\mathbf{M}$.
$\det(\mathbf{M})$	Determinant of a matrix $\mathbf{M}$.
$\mathrm{rank}(\mathbf{M})$	Rank of a matrix $\mathbf{M}$.
$\mathrm{diag}(\mathbf{M})$	Diagonal matrix with the same diagonal elements as its argument $\mathbf{M}$.
$\mathrm{Diag}(x)$	Diagonal matrix whose diagonal elements are the vector x.
$\mathrm{trace}(\mathbf{M})$	Trace of a square matrix $\mathbf{M}$.
$\mathrm{Im}(\mathbf{M})$	Range of $\mathbf{M}$.
$\ker(\mathbf{M})$	Kernel of $\mathbf{M}$.
$\mathrm{vect}(\mathbf{M})$	Stacks the columns of $\mathbf{M}$ in a long column vector.
$\mathbf{M}^+$	Moore-Penrose pseudo-inverse of $\mathbf{M}$.
$\mathbf{I}$	Identity operator or identity matrix of appropriate dimension. $\mathbf{I}_n$ if the dimension is not clear from the context.
$\langle\cdot,\cdot\rangle$	Euclidian inner product.
$\|\cdot\|$	Associated norm.
$\|\cdot\|_p$	$p \geq 1$, ℓ_p-norm of a signal.
$\|\cdot\|_0$	ℓ_0 quasi-norm of a signal; number of non-zero elements.
$\|\cdot\|_{\mathrm{TV}}$	Discrete total variation (semi)norm.
$\overline{\nabla}$	Discrete gradient of an image.
$\overline{\mathrm{div}}$	Discrete divergence operator (adjoint of $\overline{\nabla}$).
$\vert\vert\vert\cdot\vert\vert\vert$	Spectral norm for linear operators.
$\|\cdot\|_{\mathrm{F}}$	Frobenius norm of a matrix.

$\otimes$	Tensor product.
$\odot$	Entrywise (Hadamard) product.

Random variables and vectors

$\varepsilon \sim \mathcal{N}(\mu, \mathbf{\Sigma})$	ε is normally distributed with mean μ and covariance $\mathbf{\Sigma}$.
$\varepsilon \sim \mathcal{N}(\mu, \sigma^2)$	ε is additive white Gaussian with mean μ and variance σ^2.
$\varepsilon \sim \mathcal{P}(\lambda)$	ε is Poisson distributed with intensity (mean) λ.
$\mathbb{E}[.]$	Expectation operator.
Var[.]	Variance operator.
$\phi(\varepsilon; \mu, \sigma^2)$	Normal probability density function of mean μ and variance σ^2.
$\Phi(\varepsilon; \mu, \sigma^2)$	Normal cumulative distribution of mean μ and variance σ^2.

Foreword

Often, nowadays, one addresses the public understanding of mathematics and rigor by pointing to important applications and how they underpin a great deal of science and engineering. In this context, multiple resolution methods in image and signal processing, as discussed in depth in this book, are important. Results of such methods are often visual. Results too can often be presented to the layperson in an easily understood way. In addition to those aspects that speak powerfully in favor of the methods presented here, the following is worth noting. Among the most cited articles in statistics and signal processing, one finds works in the general area of what we cover in this book.

The methods discussed in this book are essential underpinnings of data analysis and are of relevance to multimedia data processing and to image, video, and signal processing. The methods discussed here feature very crucially in statistics, in mathematical methods, and in computational techniques.

Domains of application are incredibly wide, including imaging and signal processing in biology, medicine, and the life sciences generally; astronomy, physics, and the natural sciences; seismology and land use studies as indicative subdomains from geology and geography in the earth sciences; materials science, metrology, and other areas of mechanical and civil engineering; image and video compression, analysis, and synthesis for movie and television; and so on.

There is a weakness, though, in regard to well-written available works in this area: the very rigor of the methods also means that the ideas can be very deep. When separated from the means to apply and to experiment with the methods, the theory and underpinnings can require a great deal of background knowledge and diligence, and study too, in order to grasp the essential material.

Our aim in this book is to provide an essential bridge between theoretical background and easily applicable experimentation. We have an additional aim, namely that coverage is as extensive as can be, given the dynamic and broad field with which we are dealing.

Our approach, which is wedded to theory and practice, is based on a great deal of practical engagement across many application areas. Very varied applications are used for illustration and discussion in this book. This is natural, given how ubiquitous the wavelet and other multiresolution transforms have become. These transforms have become essential building blocks for addressing problems across most of data, signal, image, and indeed information handling and processing. We can characterize our approach as premised on an *embedded systems* view of how and where wavelets and multiresolution methods are to be used.

Each chapter has a section titled "Guided Numerical Experiments" complementing the accompanying description. In fact, these sections independently provide the reader with a set of recipes for a quick and easy trial and assessment of the methods presented. Our bridging of theory and practice uses openly accessible and freely available, as well as very widely used, Matlab toolboxes. In addition, IDL is used, and all code described and used here is freely available.

The scripts that we discuss in this book are available online (www.SparseSignalRecipes.info) together with the sample images used. In this form the software code is succinct and easily shown in the text of the book. The code caters to all commonly used platforms: Windows, Macintosh, Linux, and other Unix systems.

In this book we exemplify the theme of *reproducible research*. Reproducibility is at the heart of the scientific method and all successful technology development. In theoretical disciplines, the gold standard has been set by mathematics, where formal proof in principle allows anyone to reproduce the cognitive steps leading to verification of a theorem. In experimental disciplines, such as biology, physics, or chemistry, for a result to be well established, particular attention is paid to experiment replication. Computational science is a much younger field than mathematics, but already of great importance. By reproducibility of research here it is recognized that the outcome of a research project is not just the publication, but rather the entire environment used to reproduce the results presented, including data, software, and documentation. An inspiring early example was Don Knuth's seminal notion of literate programming that he developed in the 1980s in order to ensure trust or even understanding for software code and algorithms. In the late 1980s Jon Claerbout, at Stanford, used the Unix Make tool to guarantee automatic rebuilding of all results in a paper. He imposed on his group the discipline that all research books and publications originating from his group be completely reproducible.

In computational science a paradigmatic end product is a figure in a paper. Unfortunately it is rare that the reader can attempt to rebuild the authors' complex system in an attempt to understand what the authors might have done over months or years. By providing software and data sets coupled to the figures in this book, we enable the reader to reproduce what we have here.

This book provides both a means to access the state of the art in theory and to experiment through the software provided. By applying in practice the many cutting-edge signal processing approaches described here, the reader will gain a great deal of understanding. As a work of reference we believe that this book will remain invaluable for a long time to come.

The book is aimed at graduate-level study, advanced undergraduate level, and self-study. Its readership includes whoever has a professional interest in image and signal processing, to begin with. Additionally the reader is a domain specialist in data analysis in any of a very wide swath of applications who wants to adopt innovative approaches in his or her field. A further class of reader is interested in learning all there is to know about the potential of multiscale methods and also in having a very complete overview of the most recent perspectives in this area. Another class of reader is undoubtedly the student, an advanced undergraduate project student, for example, or a PhD student, who needs to grasp theory and application-oriented understanding quickly and decisively in quite varied application fields, as well as in statistics, industrially oriented mathematics, electrical engineering, and elsewhere.

The central themes of this book are *scale*, *sparsity*, and *morphological diversity*. The term *sparsity* implies a form of parsimony. *Scale* is synonymous with *resolution*. *Morphological diversity* implies use of the most appropriate morphological building blocks.

Colleagues we would like to acknowledge include Bedros Afeyan, Nabila Aghanim, Albert Bijaoui, Emmanuel Candès, Christophe Chesneau, David Donoho, Miki Elad, Olivier Forni, Gabriel Peyré, Bo Zhang, Simon Beckouche, Gitta Kutyniok, Julien Girard, and Hugh Garsden. We would like to particularly acknowledge Jérôme Bobin and Jeremy Rapin who contributed to the blind source separation chapter. We acknowledge joint analysis work with the following, relating to images in Chapter 1: Will Aicken, Kurt Birkle, P.A.M. Basheer, Adrian Long and and Paul Walsh. For their contributions to the new chapter on 3-D analysis, we would like to acknowledge François Lanusse and Arnaud Woiselle. François Lanusse is also thanked for joint analysis work in regard to wavelets on the ball or sphere. The cover was designed by Aurélie Bordenave (www.aurel-illus.com). We thank her for this work.

The following covers the major changes in this second edition. Chapter 5, formerly titled "The Ridgelet and Curvelet Transforms," has been renamed "Multiscale Geometric Transforms." Chapter 3 elaborates further on the starlet wavelet transform.

Chapter 7 is a greatly rewritten state-of-the-art description of required computation. Research has been very active in this area, since the first edition. This chapter has been reorganized and rewritten in order to take account of this. It has been organized such that the reader starts with problems, and therefore the associated algorithms, that are the most simple and then progresses to those that are more sophisticated and complex. Algorithms are focused on, over and above theory. As elsewhere, Matlab code that implements these algorithms is made available.

Chapter 9 contains new work on nonnegative matrix factorization. Chapter 11 relating to three-dimensional data is new. It covers 3-D wavelets, 3-D ridgelets and beamlets, and 3-D curvelets. An application to inpainting of magnetic resonance imaging (MRI) data of the brain is described. Chapter 12 covers new work for data on the sphere, as typifies geolocated data. This chapter includes new applications in fusion physics and, in cosmology, for the cold dark matter simulation.

Chapter 13, on compressed sensing covers the recent evolution of theory in this domain. It has also been updated in regard to the European Space Agency's Herschel mission. Recent results on real data—a compressed sensing study on Herschel data of the galaxy, NGC6946 – are presented.