MATRIX ITERATIVE ANALYSIS

by Richard S. Varga

Matrix Iterative Analysis

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PREFACE

With the growth in speed and complexity of modern digital computers has come an increase in the use of computers by those who wish to find or approximate the solutions of partial differential equations in several varibles. This increasing use of computers for such problems has correspondingly interested many mathematicians in the underlying theory for that smaller branch of numerical analysis concerned with the efficient solution of matrix problems arising from discrete approximations to elliptic partial differential equations. This current interest has generated sufficient important mathematical contributions to warrant a survey of this mathematical theory. Accordingly, our first major aim is to survey the basic results pertaining to this topic.

The basic material for this book is closely aligned with modern computing methods. The author was fortunate to have been associated with the Mathematics Group of the Bettis Atomic Power Laboratory where very large matrix problems (of order 20,000 in two dimensions!) are solved on fast computers in the design of nuclear reactors. This valuable experience, greatfully acknowledged by the author, showed that present usage of computers to solve large scale elliptic partial differential equations is almost exclusively confined to *cyclic* iterative methods. In contrast, *non-cyclic* methods—such as Southwell's relaxation method, which has been widely used for many years on desk calculators—have received far less use on computers. Accordingly, we shall look only into the mathematical theory of cyclic iterative methods. Interestingly enough, the basis for the analysis of such modern cyclic iterative methods can be traced back to fundamental research by Perron and Frobenius on non-negative matrices, and our first aim is more nearly to survey the basic results on cyclic iterative methods, using the Perron-Frobenius theory as a basis.

The material given here is intended as a text for first year graduate students in mathematics. This material, an outgrowth of courses given at

the University of Pittsburgh (1957–58) and Case Institute of Technology (1960–61), assumes familiarity with basic knowledge in matrix and linear algebra. For the most part, the material makes unstinting use of familiar matrix results and notations. But the author has not hesitated to introduce nonalgebraic items. For example, the useful notion of a *directed graph* is introduced early in Chapter 1 to help clarify the concept of irreducibility of a matrix. Later, it plays a useful role in deriving matrix properties of discrete approximations to elliptic partial differential equations. Similarly, the classical notion of *Padé rational approximations* of functions is used in Chapter 8 as a basis for generating numerical methods for parabolic partial differential equations.

To serve as an aid to the instructor using this material in the classroom, exercises are included after each section of a chapter. These often theoretically extend the material in the section. Occasionally, the exercises are numerical in nature; and even limited numerical experience will be of value to the reader.

A brief summary of the contents follows: Chapter 1 introduces vector and matrix norms, as well as directed graph theory and diagonally dominant matrices. Chapter 2 discusses the Perron-Frobenius theory of nonnegative matrices. The next three chapters are basically concerned with the analysis of variants of the successive overrelaxation (SOR) iterative method. Chapter 6 presents several viewpoints on the derivation of difference approximations to elliptic differential equations, including the Ritz form of the variational method. Chapter 7 is devoted to variants of the alternating direction implicit (ADI) methods. Chapter 8 investigates parabolic partial differential equations and obtains an association between the nature of basic iterative methods and parabolic partial differential equations. Chapter 9 treats theoretically the practical problem of the estimation of optimum iteration parameters. Finally, the two appendices contain numerical results.

While writing this manuscript, I have received valuable suggestions from many unselfish friends, colleagues, and students. To all, I give my sincere thanks. I especially want to thank Professors Garrett Birkhoff, David Young, George Forsythe, and Alston Householder and Raymond Nelson for their encouragement and helpful comments on early manuscripts. I also wish to thank R. Laurence Johnston and Louis A. Hageman, who carried out the numerical calculations; Harvey S. Price, who diligently checked all the exercises; and Martin Levy and William Roudebush, who carefully read the manuscript. Finally, sincere thanks are due to Mrs. Sarolta Petro, who, with great patience and fortitude, typed all the versions of the manuscript.

R.S.V.

GLOSSARY OF SYMBOLS

$\mathbf{x}^T$	transpose of $\mathbf{x}$, 7
$\mathbf{x}^*$	conjugate transpose of $\mathbf{x}$, 7
$\|\| \mathbf{x} \|\|$	Euclidean norm of $\mathbf{x}$, 8
$\rho(A)$	spectral radius of A, 9
$\|\| A \|\|$	spectral norm of A, 9
A^T	transpose of A, 11
A^*	conjugate transpose of A, 11
$\|\| \mathbf{x} \|\|_1$	l_1-norm of $\mathbf{x}$, 15
$\|\| \mathbf{x} \|\|_\infty$	l_∞-norm of $\mathbf{x}$, 15
$\|\| A \|\|_1$	l_1-norm of A, 15
$\|\| A \|\|_\infty$	l_∞-norm of A, 15
$\det B$	determinant of B, 26
O	null matrix, 26
$\phi(t)$	characteristic polynomial, 31
$\gamma(A)$	index of primitivity of A, 42
$\mathrm{tr}(A)$	trace of A, 44
$R(A^m)$	average rate of convergence, 62
$R_\infty(A)$	asymptotic rate of convergence, 67
$\exp(A)$	exponential of A, 87

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CHAPTER 1

MATRIX PROPERTIES AND CONCEPTS

1.1. INTRODUCTION

The title of this book, *Matrix Iterative Analysis*, suggests that we might consider here all matrix numerical methods which are iterative in nature. However, such an ambitious goal is in fact replaced by the more practical one where we seek to consider in some detail that smaller branch of numerical analysis concerned with the efficient solution, by means of iteration, of matrix equations arising from discrete approximations to partial differential equations. These matrix equations are generally characterized by the property that the associated square matrices are *sparse*, i.e., a large percentage of the entries of these matrices are zero. Furthermore, the nonzero entries of these matrices occur in some natural pattern, which, relative to a digital computer, permits even very large-order matrices to be efficiently stored. Cyclic iterative methods are ideally suited for such matrix equations, since each step requires relatively little digital computer storage or arithmetic computation. As an example of the magnitude of problems that have been successfully solved on digital computers by cyclic iterative methods, the Bettis Atomic Power Laboratory of the Westinghouse Electric Corporation had in daily use in 1960 a two-dimensional program which would treat as a special case, Laplacian-type matrix equations of order 20,000.†

The idea of solving large systems of linear equations by iterative methods is certainly not new, dating back at least to Gauss (1823). Later, Southwell (1946) and his school gave real impetus to the use of iterative methods when they systematically considered the numerical solution of

† This program, called "PDQ-4," was specifically written for the Philco-2000 computer with 32,000 words of core storage. Even more staggering is Bettis' use of a three-dimensional program, "TNT-1," which treats coupled matrix equations of order 108,000.

practical physics and engineering problems. The iterative method of *relaxation* advocated by Southwell, a *noncyclic* iterative method, was successfully used for many years by those who used either pencil and paper or desk calculators to carry out the necessary arithmetical steps, and this method was especially effective when human insight guided the entire course of the computations. With the advent of large-scale digital computers, this human insight was generally difficult to incorporate efficiently into computer programs. Accordingly, mathematicians began to look for ways of accelerating the convergence of basic *cyclic* or systematic iterative methods, methods which when initially prescribed are *not* to be altered in the course of solving matrix equations—in direct contrast with the noncyclic methods. We will concern ourselves here only with cyclic iterative methods (which for brevity we call *iterative methods*); the theory and applications of noncyclic iterative methods have been quite adequately covered elsewhere,† and these latter iterative methods generally are not used on large digital computers.

The basis for much of the present activity in this area of numerical analysis concerned with cyclic iterative methods is a series of papers by Frankel (1950), Geiringer (1949), Reich (1949), Stein and Rosenberg (1948), and Young (1950), all of which appeared when digital computers were emerging with revolutionary force. Because of the great impact of these papers on the stream of current research in this area, we have found it convenient to define *modern* matrix iterative analysis as having begun in about 1948 with the work of the above-mentioned authors. Starting at this point, our *first aim* is to describe the basic results of modern matrix iterative analysis from its beginning to the present.

We have presupposed here a basic knowledge of matrix and linear algebra theory, material which is thoroughly covered in the outstanding books by Birkhoff and MacLane (1953), Faddeeva (1959) and Bellman (1960). Thus, the reader is assumed to know, for example, what the Jordan normal form of a square complex matrix is.

Except for several isolated topics, which can be read independently, our *second aim* is to have the material here reasonably self-contained and complete. As we shall see, our development of matrix iterative analysis depends fundamentally on the early research of Perron (1907) and Frobenius (1908–12) on matrices with non-negative entries; thus, our first aim is not only to describe the basic results in this field, but also to use the Perron-Frobenius theory of non-negative matrices as a foundation for the exposition of these results. With the goal of having the material self-contained, we have devoted Chapter 2 to the Perron-Frobenius theory, although recently an excellent book by Gantmacker (1959) has also devoted a chapter to this topic.

† References are given in the Bibliography and Discussion at the end of this chapter.

Our *third aim* is to present sufficient numerical detail for those who are ultimately interested in the practical applications of the theory to the numerical solution of partial differential equations. To this end, included in Appendices A and B are illustrative examples which show the transition through the stages from problem formulation, derivation of matrix equations, application of various iterative methods, to the final examination of numerical results typical of digital computer output. Those interested in actual numerical applications are strongly urged to carry through in detail the examples presented in these Appendices. We have also included exercises for the reader in each chapter; these not only test the mastery of the material of the chapter, but in many cases allow us to indicate interesting theoretical results and extensions which have not been covered in the text. Starred exercises may require more effort on the part of the reader.

The material in this book is so organized that the general derivation of matrix equations (Chapter 6) from self-adjoint elliptic partial differential equations is not discussed until a large body of theory has been presented. The unsuspecting reader may feel he has been purposely burdened with a great number of "unessential" (from the numerical point of view) theorems and lemmas before any applications have appeared. In order to ease this burden, and to give motivation to this theory, in the next section we shall consider an especially simple example arising from the numerical solution of the Dirichlet problem showing how non-negative matrices occur naturally. Finally, the remainder of Chapter 1 deals with some fundamental concepts and results of matrix numerical analysis.

There are several important associated topics which for reasons of space are only briefly mentioned. The analysis of the effect of rounding errors and the question of convergence of the discrete solution of a system of linear equations to the continuous solution of the related partial differential equation as the mesh size tends to zero in general require mathematical tools which are quite different from those used in the matrix analysis of iterative methods. We have listed important references for these topics in the Bibliography and Discussion for this chapter.

1.2. A SIMPLE EXAMPLE

We now consider the numerical solution of the Dirichlet problem for the unit square, i.e., we seek approximations to the function $u(x, y)$ defined in the closed unit square which satisfies Laplace's equation

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = u_{xx}(x, y) + u_{yy}(x, y) = 0, \tag{1.1}$$

$$0 < x, y < 1,$$

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in the interior of the unit square. If Γ denotes the boundary of the square, then in addition to the differential equation of (1.1), $u(x, y)$ is to satisfy the Dirichlet boundary condition

$$u(x, y) = g(x, y), \qquad (x, y) \in \Gamma, \tag{1.2}$$

where $g(x, y)$ is some specified function defined on Γ. We now impose a uniform square mesh of side $h = \frac{1}{3}$ on this unit square, and we number the interior and boundary intersections (mesh points) of the horizontal and vertical line segments by means of appropriate subscripts, as shown in Figure 1. Instead of attempting to find the function $u(x, y)$ satisfying (1.1) for *all* $0 < x, y < 1$ and the boundary condition of (1.2), we seek only approximations to this function $u(x, y)$ at just the interior mesh points of the unit square. Although there are a number of different ways (Chapter 6) of finding such approximations of $u(x, y)$, one simple procedure begins by expanding the function $u(x, y)$ in a Taylor's series in two variables. Assuming that $u(x, y)$ is sufficiently differentiable, then

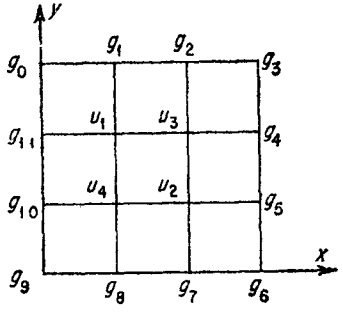

Figure 1

$$u(x_0 \pm h, y_0) = u(x_0, y_0) \pm hu_x(x_0, y_0) + \frac{h^2}{2}u_{xx}(x_0, y_0) \pm \frac{h^3}{3!}u_{xxx}(x_0, y_0) + \frac{h^4}{4!}u_{xxxx}(x_0, y_0) \pm \cdots, \tag{1.3}$$

$$u(x_0, y_0 \pm h) = u(x_0, y_0) \pm hu_y(x_0, y_0) + \frac{h^2}{2}u_{yy}(x_0, y_0) \pm \frac{h^3}{3!}u_{yyy}(x_0, y_0) + \frac{h^4}{4!}u_{yyyy}(x_0, y_0) \pm \cdots, \tag{1.3'}$$

where the point (x_0, y_0) and its four neighboring points $(x_0 \pm h, y_0)$, $(x_0, y_0 \pm h)$ are points of the closed unit square. We find then that

$$\begin{aligned} \frac{1}{h^2}\{&u(x_0 + h, y_0) + u(x_0 - h, y_0) + u(x_0, y_0 + h) \\ &+ u(x_0, y_0 - h) - 4u(x_0, y_0)\} \\ &= \{u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0)\} \\ &\quad + \frac{h^2}{12}\{u_{xxxx}(x_0, y_0) + u_{yyyy}(x_0, y_0)\} + \cdots. \end{aligned} \tag{1.4}$$

From (1.1), the first term of the right side of (1.4) is zero, and if we neglect terms with coefficients h^2 or higher, we have approximately

$$(1.4') \qquad u(x_0, y_0) \doteq \tfrac{1}{4}\{u(x_0 + h, y_0) + u(x_0 - h, y_0) + u(x_0, y_0 + h) + u(x_0, y_0 - h)\}.$$

If we let

$$u_1 \equiv u(\tfrac{1}{3}, \tfrac{2}{3}), \quad u_2 \equiv u(\tfrac{2}{3}, \tfrac{1}{3}), \quad u_3 \equiv u(\tfrac{2}{3}, \tfrac{2}{3}), \quad \text{and} \quad u_4 \equiv u(\tfrac{1}{3}, \tfrac{1}{3}),$$

and similarly g_0 is the value of the specified function $g(x, y)$ at the origin $x = y = 0$, etc., we now define respectively approximations w_i for the values u_i, $1 \leq i \leq 4$, by means of (1.4'):

$$(1.5) \qquad \begin{aligned} w_1 &= \tfrac{1}{4}(w_3 + w_4 + g_1 + g_{11}), \\ w_2 &= \tfrac{1}{4}(w_3 + w_4 + g_5 + g_7), \\ w_3 &= \tfrac{1}{4}(w_1 + w_2 + g_2 + g_4), \\ w_4 &= \tfrac{1}{4}(w_1 + w_2 + g_8 + g_{10}), \end{aligned}$$

which are then four linear equations in the four unknowns w_i, each equation representing the approximate value of the unknown function $u(x, y)$ at an interior mesh point as an average of the approximate values of $u(x, y)$ at neighboring mesh points. In matrix notation, (1.5) can be written as

$$(1.5') \qquad A\mathbf{w} = \mathbf{k},$$

where

$$(1.6) \quad A = \begin{bmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, \quad \text{and} \quad \mathbf{k} = \frac{1}{4}\begin{bmatrix} g_1 + g_{11} \\ g_5 + g_7 \\ g_2 + g_4 \\ g_8 + g_{10} \end{bmatrix}.$$

Here, $\mathbf{k}$ is a vector whose components can be calculated from the known boundary values g_i. Now, it is obvious that the matrix A can be written as $I - B$, where

$$(1.7) \qquad B = \frac{1}{4}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Evidently, both the matrices A and B are real and symmetric, and it is clear that the entries of the matrix B are all non-negative real numbers. The characteristic polynomial of the matrix B turns out to be simply

$$\phi(\mu) = \det(\mu I - B) = \mu^2(\mu^2 - \tfrac{1}{4}), \tag{1.8}$$

so that the eigenvalues of B are $\mu_1 = -\frac{1}{2}$, $\mu_2 = 0 = \mu_3$, and $\mu_4 = \frac{1}{2}$, and thus

$$\max_{1 \le i \le 4} |\mu_i| = \frac{1}{2}.$$

Since the eigenvalues ν_i of A are of the form $1 - \mu_i$, the eigenvalues of A are evidently positive real numbers, and it follows that A is a real, symmetric, and positive definite matrix. As the matrix A is nonsingular, its inverse matrix A^{-1} is uniquely defined and is given explicitly by

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 7 & 1 & 2 & 2 \\ 1 & 7 & 2 & 2 \\ 2 & 2 & 7 & 1 \\ 2 & 2 & 1 & 7 \end{bmatrix}, \tag{1.9}$$

and thus the entries of the matrix A^{-1} are all positive real numbers. We shall see later (in Chapter 6) that these simple conclusions, such as the matrix B having its eigenvalues in modulus less than unity and the matrix A^{-1} having only positive real entries, hold quite generally for matrix equations derived from self-adjoint second-order elliptic partial differential equations.

Since we can write the matrix equation (1.5′) equivalently as

$$\mathbf{w} = B\mathbf{w} + \mathbf{k}, \tag{1.10}$$

we can now generate for this simple problem our first (cyclic) iterative method, called the *point Jacobi* or *point total-step method*.† If $\mathbf{w}^{(0)}$ is an arbitrary real or complex vector approximation of the unique (since A is nonsingular) solution vector $\mathbf{w}$ of (1.5′), then we successively define a sequence of vector iterates $\mathbf{w}^{(m)}$ from

$$\mathbf{w}^{(m+1)} = B\mathbf{w}^{(m)} + \mathbf{k}, \qquad m \ge 0. \tag{1.11}$$

The first questions we would ask concern the convergence of (1.11), i.e., does each $\lim_{m \to \infty} w_j^{(m)}$ exist, and assuming these limits exist, does each limit

† Other names are also associated with this iterative method. See Sec. 3.1.

equal w_j for every component j? To begin to answer this, let

$$\varepsilon^{(m)} \equiv \mathbf{w}^{(m)} - \mathbf{w}, \qquad m \geq 0,$$

where $\varepsilon^{(m)}$ is the *error vector* associated with the vector iterate $\mathbf{w}^{(m)}$. Subtracting (1.10) from (1.11), we obtain

$$\varepsilon^{(m+1)} = B\varepsilon^{(m)},$$

from which it follows inductively that

$$\varepsilon^{(m)} = B^m\varepsilon^{(0)}, \qquad m \geq 0. \tag{1.12}$$

For any component j, it is clear that $\lim_{m\to\infty} \epsilon_j^{(m)}$ exists if and only if $\lim_{m\to\infty} w_j^{(m)}$ exists, and if these limits both exist then $\lim_{m\to\infty} w_j^{(m)} = w_j$ if and only if $\lim_{m\to\infty} \epsilon_j^{(m)} = 0$. Therefore, with (1.12), if we wish each component of the error vector to vanish in the limit, we seek conditions which insure that

$$\lim_{m\to\infty} B^m\varepsilon^{(0)} = \mathbf{0}, \tag{1.13}$$

for *all* vectors $\varepsilon^{(0)}$. But seeking conditions to insure (1.13) is equivalent to determining when

$$\lim_{m\to\infty} B^m = O, \tag{1.14}$$

where O is the null $n \times n$ matrix. This will be discussed in the next section.

1.3 NORMS AND SPECTRAL RADII

The concepts of vector norms, matrix norms, and the spectral radii of matrices play an important role in iterative numerical analysis. Just as it is convenient to compare two vectors in terms of their lengths, it will be similarly convenient to compare two matrices by some measure or norm. As we shall see, this will be the basis for deciding which of two iterative methods is more rapidly convergent, in some precise sense.

To begin with, let $V_n(C)$ be the n-dimensional vector space over the field of complex numbers C of column vectors $\mathbf{x}$, where the vector $\mathbf{x}$, its transpose $\mathbf{x}^T$, and its conjugate transpose $\mathbf{x}^*$ are denoted by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{x}^T = [x_1\, x_2 \cdots x_n], \qquad \mathbf{x}^* = [\bar{x}_1\, \bar{x}_2 \cdots \bar{x}_n],$$

where $x_1, x_2, \cdots, x_n$ are complex numbers, and $\bar{x}_i$ is the complex conjugate of x_i.

DEFINITION 1.1. Let $\mathbf{x}$ be a (column) vector of $V_n(C)$. Then,

$$\| \mathbf{x} \| \equiv (\mathbf{x}^*\mathbf{x})^{1/2} = \left(\sum_{i=1}^{n} | x_i |^2 \right)^{1/2} \tag{1.15}$$

is the *Euclidean norm* (or length) of $\mathbf{x}$.

With this definition, the following results are well known.

Theorem 1.1. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors of* $V_n(C)$, *then*

$$\| \mathbf{x} \| > 0, \quad \textit{unless } \mathbf{x} = \mathbf{0};$$

$$\textit{if } \alpha \textit{ is a scalar, then } \| \alpha\mathbf{x} \| = | \alpha | \cdot \| \mathbf{x} \|; \tag{1.16}$$

$$\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|.$$

If we have an infinite sequence $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots$ of vectors of $V_n(C)$, we say that this sequence *converges* to a vector $\mathbf{x}$ of $V_n(C)$ if

$$\lim_{m \to \infty} x_j^{(m)} = x_j, \qquad \text{for all } 1 \leq j \leq n,$$

where $x_j^{(m)}$ and x_j are respectively the jth components of the vectors $\mathbf{x}^{(m)}$ and $\mathbf{x}$. Similarly, by the convergence of an infinite series $\sum_{m=0}^{\infty} \mathbf{y}^{(m)}$ of vectors of $V_n(C)$ to a vector $\mathbf{y}$ of $V_n(C)$, we mean that

$$\lim_{N \to \infty} \sum_{m=0}^{N} y_i^{(m)} = y_i, \qquad \text{for all } 1 \leq j \leq n.$$

In terms of Euclidean norms, it then follows from Definition 1.1 that

$$\| \mathbf{x}^{(m)} - \mathbf{x} \| \to 0, \qquad m \to \infty,$$

if and only if the sequence $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \cdots$ of vectors converges to the vector $\mathbf{x}$, and similarly

$$\left\| \sum_{m=0}^{N} \mathbf{y}^{(m)} - \mathbf{y} \right\| \to 0, \qquad N \to \infty,$$

if and only if the infinite series $\sum_{m=0}^{\infty} \mathbf{y}^{(m)}$ converges to the vector $\mathbf{y}$.

Our next basic definition, which will be repeatedly used in subsequent developments, is

DEFINITION 1.2. Let $A = (a_{i,j})$ be an $n \times n$ complex matrix with eigenvalues λ_i, $1 \leq i \leq n$. Then

$$\rho(A) \equiv \max_{1 \leq i \leq n} |\lambda_i| \tag{1.17}$$

is the *spectral radius* of the matrix A.

Geometrically, if all the eigenvalues λ_i of A are plotted in the complex z-plane, then $\rho(A)$ is the radius of the smallest disk† $|z| \leq R$, with center at the origin, which includes all the eigenvalues of the matrix A.

Now, we shall assign to each $n \times n$ matrix A with complex entries a non-negative real number which, like the vector norm $\|\mathbf{x}\|$, has properties of length similar to those of (1.16).

DEFINITION 1.3. If $A = (a_{i,j})$ is an $n \times n$ complex matrix, then

$$\|A\| = \sup_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}. \tag{1.18}$$

is the *spectral norm* of the matrix A.

Basic properties of the spectral norm of a matrix, analogous to those obtained for the Euclidean norm of the vector $\mathbf{x}$, are given in

Theorem 1.2. *If A and B are two $n \times n$ matrices, then*

$$\begin{aligned} &\|A\| > 0, \text{ unless } A \equiv O, \text{ the null matrix;} \\ &\text{if } \alpha \text{ is a scalar, } \|\alpha A\| = |\alpha| \cdot \|A\|; \\ &\|A + B\| \leq \|A\| + \|B\|; \\ &\|A \cdot B\| \leq \|A\| \cdot \|B\|. \end{aligned} \tag{1.19}$$

Moreover,

$$\|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\| \tag{1.20}$$

for all vectors $\mathbf{x}$, and there exists a nonzero vector $\mathbf{y}$ in $V_n(C)$ for which

$$\|A\mathbf{y}\| = \|A\| \cdot \|\mathbf{y}\|. \tag{1.21}$$

Proof. The results of (1.19) and (1.20) follow directly from Theorem 1.1 and Definition 1.3. To establish (1.21), observe that the ratio $\|A\mathbf{x}\|/\|\mathbf{x}\|$ is unchanged if $\mathbf{x}$ is replaced by $\alpha\mathbf{x}$, where α is a scalar. Hence, we can write that

$$\|A\| = \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

† To be precise, the set of points for which $|z - a| \leq R$ is called a *disk*, whereas its subset, defined by $|z - a| = R$, is called a *circle*.

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