Mathematical Methods in Engineering

Joseph M. Powers • Mihir Sen

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MATHEMATICAL METHODS IN ENGINEERING

This text focuses on a variety of topics in mathematics in common usage in graduate engineering programs including vector calculus, linear and nonlinear ordinary differential equations, approximation methods, vector spaces, linear algebra, integral equations, and dynamical systems. The book is designed for engineering graduate students who wonder how much of their basic mathematics will be of use in practice. Following development of the underlying analysis, the book takes students step-by-step through a large number of examples that have been worked in detail. Students can choose to go through each step or to skip ahead if they desire. After seeing all the intermediate steps, they will be in a better position to know what is expected of them when solving homework assignments and examination problems, and when they are on the job. Each chapter concludes with numerous exercises for the student that reinforce the chapter content and help connect the subject matter to a variety of engineering problems. Students today have grown up with computer-based tools including numerical calculations and computer graphics; the worked-out examples as well as the end-of-chapter exercises often use computers for numerical and symbolic computations and for graphical display of the results.

Joseph M. Powers joined the University of Notre Dame in 1989. His research has focused on the dynamics of high-speed reactive fluids and on computational science, especially as it applies to verification and validation of complex multiscale systems. He has held positions at the NASA Lewis Research Center, the Los Alamos National Laboratory, the Air Force Research Laboratory, the Argonne National Laboratory, and the Chinese Academy of Sciences. He is a member of AIAA, APS, ASME, the Combustion Institute, and SIAM. He is the recipient of numerous teaching awards.

Mihir Sen has been active in teaching and in research in thermal-fluids engineering – especially in regard to problems relating to modeling, dynamics, and stability – since obtaining his PhD from MIT. He has worked on reacting flows, natural and forced convection, flow in porous media, falling films, boiling, MEMS, heat exchangers, thermal control, and intelligent systems. He joined the University of Notre Dame in 1986 and received the Kaneb Teaching Award from the College of Engineering in 2001 and the Rev. Edmund P. Joyce, C.S.C., Award for Excellence in Undergraduate Teaching in 2009. He is a Fellow of ASME.

To my parents: Mary Rita, my first reading teacher, and Joseph Leo, my first mathematics teacher – Joseph Michael Powers

To my family: Beatriz, Pradeep, Maya, Yasamin, and Shayan – Mihir Sen

Preface

Our overarching aim in writing this book is to build a bridge to enable engineers to better traverse the domains of the mathematical and physical worlds. Our focus is on neither the nuances of pure mathematics nor the phenomenology of physical devices but instead is on the mathematical tools used today in many engineering environments. We often compromise strict formalism for the sake of efficient exposition of mathematical tools. Whereas some results are fully derived, others are simply asserted, especially when detailed proofs would significantly lengthen the presentation. Thus, the book emphasizes method and technique over rigor and completeness; readers who require more of the latter can and *should* turn to many of the foundational works cited in the extensive bibliography.

Our specific objective is to survey topics in engineering-relevant applied mathematics, including multivariable calculus, vectors and tensors, ordinary differential equations, approximation methods, linear analysis, linear algebra, linear integral equations, and nonlinear dynamical systems. In short, the text fully explores linear systems and considers some effects of nonlinearity, especially those types that can be treated analytically. Many topics have geometric interpretations, identified throughout the book. Particular attention is paid to the notion of approximation via projection of an entity from a high- or even infinite-dimensional space onto a space of lower dimension. Another goal is to give the student the mathematical background to delve into topics such as dynamics, differential geometry, continuum mechanics, and computational methods; although the material presented is relevant to those fields, specific physical applications are mainly confined to some of the exercises. A final goal is to introduce the engineer to *some* of the notation and rigor of mathematics in a way used in many upper-level graduate engineering and applied mathematics courses.

This book is intended for use in a beginning graduate course in applied mathematics taught to engineers. It arose from a set of notes for such a course taught by the authors for more than 20 years in the Department of Aerospace and Mechanical Engineering at the University of Notre Dame. Students in this course come from a variety of backgrounds, mainly within engineering but also from science. Most enter with some undergraduate-level proficiency in differential and integral multivariable calculus, differential equations, vectors analysis, and linear algebra. This book briefly reviews these subjects but more often builds on an assumed elementary understanding of topics such as continuity, limits, series, and the chain rule.

As such, we often casually introduce subject matter whose full development is deferred to later chapters. For example, although one of the key features of the book is a lengthy discussion of eigensystems in Chapter 6, most engineers will already know what an eigenvalue is. Consequently, we employ eigenvalues in nearly every chapter, starting from Chapter 1. The same can be said for topics such as vector operators, determinants, and linear equations. When such topics are introduced earlier than their formal presentation, we often make a forward reference to the appropriate section, and the student is encouraged to read ahead. In summary, most beginning graduate students and advanced undergraduates will be prepared for the subject matter, though they may find occasion to revisit some trusted undergraduate texts. Although our course is only one semester, we have added some topics to those we usually cover; the instructor of a similar course should be able to omit certain topics and add others.

At a time not very far in the past, mathematics in engineering was largely confined to basic algebra and interpolation of trigonometric and logarithmic tables. Not so today! Much of engineering has come to rely on sophisticated predictive mathematical models to design and control a wide variety of devices, for example, buildings and bridges, air and ground transportation, manufacturing and chemical processes, electrical and electronic devices, and biomedical and robotic equipment. These models may be in the form of algebraic, differential, integral, or other equations. Formulation of such models is often a challenge that calls on the basic sciences of physics, chemistry, and biology. Once they are formulated, the engineer is faced with actually solving the model equations, and for that a variety of tools are of value. Our focus is on the general mathematical tools used for engineering problems but not their formulation or specific physical details. While we sporadically discuss a paradigm problem such as a mass-spring-damper, we focus more on the mathematics. The use of mathematical analysis within engineering has changed greatly over the years. Once it was the sole means to the solution of some problems, but currently engineers rely on it within numerical and experimental approaches. The use of the adjective *numerical* has also changed over the years because of the variety of ways in which computers may be used in engineering. It is in fact becoming more common and necessary for extensive mathematical manipulation to be performed to prepare the computer for efficient and accurate solution generation.

Choices have been made with regard to notation; most of the conventions we adopt are reflected in at least a portion of the literature. In a few cases, we choose to diverge slightly from some of the more common norms. When we do so, explanatory footnotes are usually included. For example, the literature has a number of conventions for the so-called dot product or scalar product between two vectors, $\mathbf{u}$ and $\mathbf{v}$. The most common is probably $\mathbf{u} \cdot \mathbf{v}$. We generally choose the more elaborate $\mathbf{u}^T \cdot \mathbf{v}$, where the T indicates a transpose operation. This emphasizes the fact that vectors are considered to be columns of elements and that to associate a scalar product of two vectors with the ordinary rules of matrix multiplication, one needs to transpose the first vector into a row of elements. Similarly, we generally take the product of a matrix $\mathbf{A}$ and vector $\mathbf{u}$ to be of the form $\mathbf{A} \cdot \mathbf{u}$ rather than the often-used $\mathbf{Au}$. And we use $\mathbf{u}^T \cdot \mathbf{A}$, while many texts simply write $\mathbf{uA}$. Unusually, we often apply the transpose notation to the so-called divergence operator, writing, for example, the divergence of a vector field $\mathbf{u}$ as $\text{div}\,\mathbf{u} = \nabla^T \cdot \mathbf{u}$ rather than as the more common $\nabla \cdot \mathbf{u}$. One could easily infer the nature of the divergence operation without the transpose, but we believe it adds unity to our notational framework. In the text, italicized letters like a will most often be used for scalars, bold lowercase letters like $\mathbf{a}$ for vectors, bold uppercase letters like $\mathbf{A}$ for tensors and operators, and $\mathbb{A}$ for sets and spaces. In general, we use ^{T} for transpose, $^-$ for complex conjugate, * for adjoint, and ^{H} for Hermitian transpose. The student also has to be aware that the same quantities written on a blackboard or paper may appear differently. Whatever the notational choices, the student should be fluent in a variety of usages in the literature; we in fact sometimes deviate from our own conventions to reinforce that our choices are not unique.

Our experience has been that engineering students learn best by exposure to examples, and a hallmark of the text is that much of the material is developed via presentation of a large number of fully worked problems, each of which generally follows a short fundamental development. The solved examples not only illustrate the points made previously but also introduce additional concepts and are thus an integral part of the text. Ultimately, mathematics is learned by doing, and for this reason we have a large number of exercises. Engineers, moreover, have a special purpose for studying mathematics: they need it to solve practical problems; some are included as exercises.

Presentation of many of our specific details has relied on modern software for symbolic computation and plotting. We encourage the reader to utilize these tools as well, as they enable exact solutions and graphics that may otherwise be impossible. The text does not provide details of particular software packages, which often change with each new version; the reader is advised to choose one or more packages, such as Mathematica, Maple, or MATLAB, and to become familiar with its usage. Exercises are included that require the use of such software. The phrase "symbolic computer mathematics" is used to mean tools such as Mathematica or Maple and "discrete computational methods" to connote tools such as MATLAB, Python, Fortran, C, or C++. The problems emphasize plotting to give a geometric overview of the results. The use of visuals or graphics to get a quick appreciation for the quality of an approximation or the behavior of a result cannot be overemphasized.

There are a number of texts on graduate-level mathematics for engineers. Mathematics applied to engineering is a vast discipline; consequently, each book has a unique emphasis. Here we have attempted to include what is actually used by researchers in our field. To be clear, though, because it is for a one-semester introductory course, many topics are left for advanced courses. Among the topics omitted or lightly covered are integral transforms, complex variables, partial differential equations, group theory, probability, statistics, numerical methods, and graph and network theory.

In closing, we express our hope that the readers of this book will find mathematics to be as beautiful and useful a subject as we have over the years. Our appreciation was nurtured by a large number of special people: family members, teachers at all levels, colleagues at home and abroad, authors from many ages, and our own students over the decades. We have learned from all of them and hope that our propagation of their knowledge engenders new discoveries from readers of this book for future generations.

Joseph M. Powers
Mihir Sen
Notre Dame, Indiana, USA

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