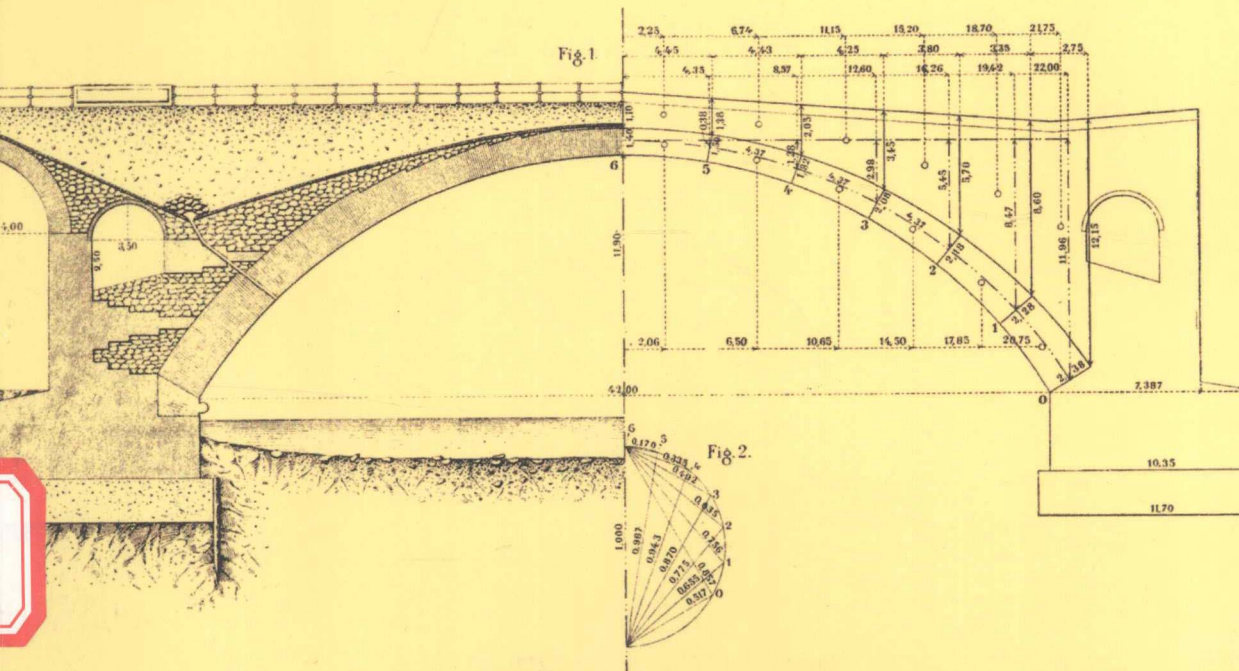


CAMBRIDGE LIBRARY COLLECTION

# ELASTIC STRESSES IN STRUCTURES

TRANSLATED FROM CASTIGLIANO'S  
THÉOREM DE L'EQUIBRE DES SYSTÈMES ÉLASTIQUES  
ET SES APPLICATIONS

ALBERTO CASTIGLIANO  
TRANSLATED BY EWART S. ANDREWS



CAMBRIDGE

# Elastic Stresses in Structures

*Translated from Castigliano's  
Théorem de l'équilibre des systèmes élastiques  
et ses applications*

ALBERTO CASTIGLIANO  
TRANSLATED BY EWART S. ANDREWS



CAMBRIDGE  
UNIVERSITY PRESS

**CAMBRIDGE**  
**UNIVERSITY PRESS**

University Printing House, Cambridge, CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.  
It furthers the University's mission by disseminating knowledge in the pursuit of  
education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781108070263](http://www.cambridge.org/9781108070263)

© in this compilation Cambridge University Press 2014

This edition first published 1919  
This digitally printed version 2014

ISBN 978-1-108-07026-3 Paperback

This book reproduces the text of the original edition. The content and language reflect  
the beliefs, practices and terminology of their time, and have not been updated.

Cambridge University Press wishes to make clear that the book, unless originally published  
by Cambridge, is not being republished by, in association or collaboration with,  
or with the endorsement or approval of, the original publisher or its successors in title.

# ELASTIC STRESSES IN STRUCTURES

TRANSLATED FROM CASTIGLIANO'S "THÉOREM DE L'EQUILIBRE  
DES SYSTÈMES ÉLASTIQUES ET SES APPLICATIONS"

BY

EWART S. ANDREWS, B.Sc., ENG. (LOND.)

CONSULTING ENGINEER; MEMBER OF COUNCIL OF CONCRETE INSTITUTE  
LECTURER IN THE ENGINEERING DEPARTMENT OF GOLDSMITHS'  
COLLEGE, NEW CROSS, LONDON, S.E.

WITH 15 FOLDING PLATES, CONTAINING 109 DIAGRAMS

LONDON  
SCOTT, GREENWOOD & SON  
(E. GREENWOOD)

8 BROADWAY, LUDGATE, E.C. 4

1919

## AUTHOR'S PREFACE.

THIS book contains the theory of elastic stresses in structures explained in accordance with a new method, based upon theorems which are either quite new or little known; incorporated in this theory will be found the mathematical theory of elastic solids, considered particularly from the standpoint of the strength of materials.

We believe that the time has arrived for introducing into our courses of instruction this scientific method of treating the strength of materials and thus abandoning the older methods which Lamé has fairly characterised as “neither scientific nor empirical, serving only to hinder the approach of true science”.

We will now give some historical notes of the discovery of the theorems of which we make repeated use in this book.

These theorems are the following three:—

1. That of the differential coefficients of internal work, 1st part.
2. That of the differential coefficients of internal work, 2nd part.
3. That of the least internal work.

The first has already been employed by the celebrated English astronomer Green, but only for a particular problem; it has not been enunciated and proved in a general way such as we give here.

The second is the converse of the first, and we believe that it was first enunciated and proved in our

thesis for the diploma of engineering at Turin ; we have given it at greater length in our memoir entitled "Nuova teoria intorno all' equilibrio dei sistemi elastici," published in the Transactions of the Academy of Science of Turin in 1875.

The third theorem can be regarded as a corollary of the second ; but as in some other problems of maxima and minima, it has been partly known several years before the discovery of the principal theorem.

In the year 1818 Captain Vène, of the French Engineer Corps, enunciated a principle which was absolutely incorrect under the conditions to which he wished to apply it, but which, by one of those peculiar combination of circumstances of which science presents several examples, was destined to lead later to the discovery of the theorem of least work.

After this first step, the distinguished scientists, MM. A. Cournot, Pagani, Mossotti, A. Dorna, and General L. F. Ménabréa investigated the question. The last-mentioned gave the name "principle of elasticity" to the theorem of least work, and made it the subject of his researches, in a first memoir presented in 1857 to the Academy of Science of Turin, later in a second presented in 1858 to the Academy of Science of Paris, and again in a third submitted in 1868 to the Turin Academy. Since, however, the proofs given by M. Ménabréa were not exact, the "principle of elasticity" was not accepted by the greater number of the authorities, and some of them published memoranda to show the fallacy of it. It was not until 1873 that we gave, in our above-mentioned thesis, the first rigorous proof, in a form which appeared to us clear and exact, of the theorem of least work. Afterwards, in our thesis of 1875, we demonstrated that the theorem of least work is only a corollary of that of the differential coefficients of internal work.

We can thus state that the present book, comprising the complete theory of elastic stresses in structures, of which the mathematical theory of the elasticity of solid bodies comprises only a chapter, is wholly based on the theorems of the differential coefficients of internal work. As our object is not only to expound a theory, but further to show its advantages of brevity and simplicity in practical applications, we have solved, according to the new method, the greater number of the general problems dealt with in courses on the strength of materials, and have added several numerical examples for the calculation of the stresses in the more important types of structure. Each of these examples is, so to speak, a particular application of the theory to one of these structures, but in order to give our book a more practical value we have put each example in the form which would arise in an actual design, in order to justify the dimensions of the principal members.

Moreover, since in our view these examples ought to serve as a model for similar calculations, we have always examined several cases of loading and taken account of the effects due to temperature variations; that is to say, that for each assumed load we have determined the stresses occurring at different temperatures. By taking account of these circumstances and by following the new method of calculation, which permits of the solution of all questions on the stresses in elastic structures without the introduction of any arbitrary assumption, we have the advantage of being able to adopt higher working stresses; for one of the causes which often compels, in practice, the adoption of small values for these stresses, is the imperfection of the principles upon which the calculation of the stresses is based; and another reason for the adoption of low working stresses is that we cannot take account of all the circumstances liable to cause these stresses to increase. With regard to the calculations, we may say

that they are but a little longer than in methods ordinarily followed, and moreover, they can nearly always be materially shortened by neglecting some terms having small influence on the result.

In conclusion, we have heartily to thank M. Louis Reeb, Permanent Way Superintendent of the Northern Italian Railway, who has kindly undertaken the proof revision.



## TRANSLATOR'S PREFACE.

CASTIGLIANO'S work is referred to as a classic in the leading text-books dealing with the advanced Theory of Structures, but few students in modern times have had an opportunity of studying it, since, as far as we have been able to ascertain, there is no previous translation in English, and the copy in French is out of print and very scarce.

With the development in practical design of methods of calculation involving considerations of internal work arising in "Higher Structures," the work of Castigliano becomes of fresh interest to engineers. Although the book is now practically forty years old, it is surprising that it is by no means out of date; the reason for this is that though practice may vary, principles are almost invariable. Castigliano's work gives us the most complete analysis of the theory of elasticity applied to the determination of stresses in structures that we have yet met, and we believe that it deserves to receive a close study by all engineers and students who wish to follow the logical development of structural theory and its application to practical design.

In view of the very large number of tabulated numerical values and of the resulting labour involved in translating these values from metric into British units, it has been decided to preserve the metric units; it is thought that this does not detract from the value of the book in demonstrating methods of design for British designers. In order, however, to bring the

notation into more modern and standardised form, the mnemonic notation prepared by the Concrete Institute has been followed as far as possible.

The author's thanks are due to Mr. C. Paice of the Examining Staff of H.M. Patent Office for assistance in the troublesome task of reading the proofs, and to Mr. C. Wyndham Hulme, Librarian of H.M. Patent Office, for courtesy in giving special access to the copy of M. Castigliano's book in the Patent Office Library.

EWART S. ANDREWS.

ROLLS CHAMBERS, 89 CHANCERY LANE,  
LONDON, W.C. 2, *July*, 1918.

## FOLDING PLATES

(Following Index).

- Plate I. Figures 1 to 16.  
,, II. Figures 1 to 22.  
,, III. Figures 1 to 10.  
,, IV. Figures 1 to 16, Bending Moment Diagrams.  
,, V. Figures 1 to 4, Arched Roof Truss without Tie-rods.  
,, VI. Figures 1 to 4, Arched Roof Truss with single Tie-rod.  
,, VII. Figures 1 to 5, Roof Truss of Polonceau type.  
,, VIII. Figures 1 to 16, Arched Roof Truss with several Tie-rods.  
,, IX. Figures 1 to 7, Arched Roof Truss with several Tie-rods.  
,, X. Figures 1 and 11, Arched Roof Truss with several Tie-rods.  
,, XI. Figures 2 to 10, Arched Roof Truss with several Tie-rods.  
,, XII. Figures 1 to 5, Arch Bridges with flat and rounded Ends.  
,, XIII. Figures 1 to 4, Arch Bridge with flat Ends; Arch Bridge with rounded  
Ends.  
,, XIV. Figures 1 to 3, Masonry Arch Bridge over River Oglio.  
,, XV. Figures 1 to 4, Stone Bridge over River Doire.

## NOTATION.

The following is the general notation adopted throughout the book; whenever additional notation is necessitated, its explanation is given in the text:—

A	= Area of cross section.
B	= Bending moment.
C	= Constant.
E	= Elastic modulus (Young's modulus).
F	= Force (general).
F	= Coefficient of shear stress in beam formulæ.
G	= Glide, shear or rigidity modulus.
H	= Horizontal thrust in arches.
I	= Moment of inertia (about a line).
$I_p$	= Polar moment of inertia.
M	= Moment (general).
M	= Modulus of section.
$M_1$	= First moment of a force.
	= First moment about N.A. of area above N.A. in shear in beams.
N.A.	= Neutral axis.
N	= Node.
P	= Pressure or thrust.
R	= Radius of an arch.
R	= Resultant force.
R	= Reaction.
S	= Shearing force.
T	= Tensile force.
$T_M$	= Twisting moment.
V	= Volume.
W	= Weight.
$W_i$	= Internal work.
X, Y, Z	= Components of a force parallel to the corresponding axes.
a	= An arm or distance (general).
a	= Length of prismatic body of cylinder.
a	= Breadth of a rectangle.
a	= Semi-axis of an ellipse.
b	= Semi-axis of an ellipse.
b	= Depth of a rectangle.
c	= Compressive stress.
d	= Diagonal length.
d	= Depth of a beam (general).
$d_n$	= Deflection.
e	= Eccentricity of load.

$f$	= Force per unit area.
$g$	= Gyration radius.
$l$	= Length (general). = Span of a beam or arch.
$n$	= A number. = Neutral axis depth from compressed edge of beam.
$r$	= Rise of an arch.
$r$	= Radius or radial length.
$s$	= Shear stress.
$t$	= Tensile stress.
$t^{\circ}$	= Temperature.
$u$	} = Displacements (p. 36).
$v$	
$w$	
$w$	= Weight or load per unit length.
$x$	= Extension.
$x, y, z$	= co-ordinates of a point.
$\alpha$	} = Angles.
$\beta$	
$\gamma$	
$\epsilon$	= $\frac{EA}{l}$ .
$\epsilon$	= A coefficient (p. 40).
$\xi$	= Increment of $x$ .
$\eta$	= Increment of $y$ .
$\zeta$	= Increment of $z$ .
$\lambda$	= Increment of $l$ .
$\rho$	= Increment of $r$ .
$\kappa$	= Compressive strain.
$\sigma$	= Shear strain.
$\tau$	= Tensile strain.
$\nu$	= Dilatation or volume strain.
$\Pi$	= Poisson's ratio. [Lateral contraction to longitudinal extension.]
$\pi$	= Circle function 3.14159
$\theta$	= Angle of torsion.
$\Theta$	} = Angles generally.
$\Phi$	

# CONTENTS.

NOTATION . . . . .	PAGE xix
--------------------	-------------

## PART I. THEORY.

### CHAPTER I.

FRAMED STRUCTURES . . . . .	3
-----------------------------	---

Definition—Experimental law—Internal work of a rod stretched or compressed along its axis—Properties of framed structures—Strain of framed structures—Equations for the equilibrium of framed structures—Simplification of equations—Determination of the stresses in the bars—Notes on absolutely rigid structures—Internal work as a function of the external forces—Theorem of the differential coefficient of internal work—Theorem of least work—Cases in which this theorem is not applicable—Structures in which stresses occur before external loads are applied—Temperature stresses—The principle of superposition.

### CHAPTER II.

ELASTIC STRUCTURES . . . . .	29
------------------------------	----

Definitions and fundamental principles—The principle of superposition—Property of internal work—Theorems of the differential coefficients of internal work and extension thereof—Corollary—Theorem of least work.

### CHAPTER III.

GENERAL EQUATIONS OF THE ELASTIC EQUILIBRIUM OF A SOLID . . . . .	36
---	----

Strain of a body—Unit tensile strain—Shear—Internal work of a very small parallelepiped—Stresses on the faces of the parallelepiped—Variation of stresses from point to point in a solid—Components of the stresses exerted upon a small face inclined to the axis—Simplification of stress equations when the body possesses particular elastic properties—Equations of condition relative to the surface—Differential relations between the normal and shear strains—Body of any shape acted over its whole surface by a normal and uniform pressure—Prism or cylinder stretched by forces uniformly distributed over its bases and directed along its axis—Torsion of homogeneous cylinders

or prisms—Torsion of homogeneous prism with varying elastic properties having an elliptic section—Torsion of homogeneous prism having at each point a plane of symmetry parallel to the ends, or three planes of symmetry one of which is parallel to the ends—Torsion of prism of rectangular section—Torsion of prism of any cross-section—Torsion of prism of elliptic section—Bending of homogeneous prisms or cylinders—Determination of constants—Application to rectangular prism—Application to elliptic prism—Form of prism or cylinder after bending—Prisms subjected to different kinds of forces—Equations of strength or stability—Equations of strength deduced from maximum strains—Equations of strength deduced from maximum stresses.

## CHAPTER IV.

## APPROXIMATE APPLICATIONS . . . . . 103

Experimental law—Long slender bodies curved in any manner—Relations between resultant shear forces and bending moments—Formula for internal work—Strains in long slender bodies—Smallness of internal work due to shear when there is no torsion—Curved form of sections after strain due to normal stresses only—Definitions and notation for structures that usually occur in practice—Transformation of one of the differential equations of elastic equilibrium of solid bodies—Application to solid bodies of breadth small compared with their depth—Curved surface assumed by these bodies after bending—Application of formulæ to simplest and most usual sections.

## CHAPTER V.

## THEORY OF LATTICE GIRDERS . . . . . 128

Definitions—First, second, and third cases of a pin-jointed lattice girder—Ordinary lattice girder—Internal work of lattice girders.

## CHAPTER VI.

## FORMULÆ FOR THE INTERNAL WORK OF DIFFERENT SOLID BODIES . . . . . 140

Cable or prismatic bar pin-jointed at its ends—Prismatic body loaded uniformly along its whole length and loaded in any manner at its ends—Previous case in which the centre line is bent to a circular arc having a very small rise in proportion to its chord—Solid body of constant section, loaded continuously throughout its length and having its centre line following any plain curve.

## CHAPTER VII.

## THEORY OF STRAIGHT BEAMS . . . . . 149

Definitions—Cantilever—Beam fixed at one end and simply supported at the other—Beam fixed at both ends—Beam simply supported at its ends—Simply supported beam the ends of which are not subjected to any normal force—Beams continuous over several supports; isolated loads; uniformly distributed load—Practical formulæ for design of single-span bridges; diagrams of bending moment and shear for uniform rolling loads—Isolated rolling load—Bending moment and shear diagrams for iron bridges—Deflection at the centre of a single-span bridge carrying a uniformly distributed load—Iron bridges composed of continuous beams.

## CHAPTER VIII.

PAGE

## THEORY OF COLUMNS . . . . . 184

Curve assumed by a vertical beam when loaded at the top—Integration of equation for homogeneous columns of constant section—Discussion of equation when shear at end is zero—Application to fixed ends—Third special case—Discussion of equation when end bending moment is zero.

## CHAPTER IX.

## THEORY OF CURVED RIBS AND ARCHES . . . . . 195

Strains in structure curved in one plane—Cases in which the formulæ are inapplicable—Application to an arch with rounded ends—Application to arches with flat ends—Application to arches of any form carrying a continuous load—Line of pressure.

## CHAPTER X.

## THEORY OF COMPOSITE STRUCTURES . . . . . 214

Determination of unknown stresses—Change of the unknowns—Calculations for symmetrical structures—Use of the principle of superposition for the study of composite structures—Proof of the small influence on the stresses in elastic structures of the terms due to shear—Theorem of the effects of temperature variation—Simple graphical constructions to replace numerical calculations.

## CHAPTER XI.

## IMPERFECTLY ELASTIC STRESSES SUCH AS MASONRY ARCHES 233

Some notes on imperfectly elastic structures.

## PART II. APPLICATION TO PRACTICAL EXAMPLES.

## CHAPTER XII.

## STUDY OF A BEAM STRENGTHENED BY TWO TIE-RODS AND A CAST-IRON STRUT . . . . . 237

General data—Calculation of tension in the tie-rods—Maximum stress per sq. metre in the tie-rods and beam—Effect of temperature variation.

## CHAPTER XIII.

## STUDY OF A BEAM STRENGTHENED BY THREE TIE-BARS AND TWO CAST-IRON STRUTS . . . . . 242

General data—Calculation of the tension in the tie-bars—Maximum stress per sq. metre in the tie-bars and beam—Temperature effects—Conclusion.

## CHAPTER XIV.

## STUDY OF A BEAM STRENGTHENED BY THREE TIE-BARS AND TWO CAST-IRON STRUTS . . . . . 248

General data—Calculation of the tension in the tie-bars—Maximum stress per sq. metre in the tie bars and beam—Temperature effects—Conclusion.



	PAGE
CHAPTER XV.	
STUDY OF A ROOF WITH IRON TRUSSES WITHOUT TIE-RODS	254
<p>General data—Loads—Principles and general formulæ for the case of the superload on the left-hand half of the truss, neglecting the dead load—Determination of the value of the unknowns—Line of pressure and maximum stress per sq. metre in the iron, taking account of the dead load and of the superload both on the right and left of the crown—The same taking the dead load into account and assuming that the superload is only on the left-hand half of the arch—Temperature effects.</p>	
CHAPTER XVI.	
STUDY OF AN ARCHED ROOF-TRUSS WITH A SINGLE TIE-ROD	267
<p>General data—Loads—Principles and general formulæ for the case of the superload over the whole arch—Determination of the value of the unknown T—Line of pressure and maximum stresses per sq. metre in the tie-rod and arch—Examination of the case in which only the left-hand half of the truss is loaded—Temperature effects.</p>	
CHAPTER XVII.	
STUDY OF A ROOF-TRUSS OF THE POLONCEAU TYPE . . .	274
<p>General data—Loads—Principles and general formulæ for the superload on the whole truss—Stresses per sq. metre in tie-rods and strut—Line of pressure and maximum stresses per sq. metre in the rafters—Examination of the case in which the superload is on one side only of the truss and the dead load is left out of account—The case in which account is taken of the dead load on the whole truss and the superload on the left-hand side only—Temperature stresses.</p>	
CHAPTER XVIII.	
STUDY OF AN ARCHED ROOF-TRUSS WITH SEVERAL TIE-RODS	284
<p>General data—Loads—Notation and general formulæ for the case in which the arch is fully loaded—Calculation of the tensions <math>T_1</math> and <math>T_4</math>—Stress per sq. metre in the tie-rods—Line of pressure and maximum stresses for the arch—Dimensions to be adopted—Examination of the case in which the superload is on the left-hand half of the arch only, the dead load not being taken into account—Examination of the case in which the dead load is taken into account and the superload is taken as acting on the left-hand side only—Temperature effects.</p>	
CHAPTER XIX.	
STUDY OF AN ARCHED ROOF-TRUSS WITH SEVERAL TIE-BARS	292
<p>General data—Loads—Notation and general formulæ—Calculation of the tensions <math>T_1</math> and <math>T_4</math>—Maximum stress per sq. metre in tie-bars—Line of pressure of the arch and maximum stresses per sq. metre.</p>	
CHAPTER XX.	
STUDY OF AN ARCHED ROOF-TRUSS WITH SEVERAL TIE-BARS (CRESCENT TRUSS) . . . . .	298
<p>General data—Loads—Calculations for the superload on both sides of the truss; notation and general formulæ—Determination of the unknowns <math>T_2, T_3, T_4</math>—Stresses per sq. metre in the tie-bars and in the arch—</p>	