COLLEGE ALGEBRA

SECOND EDITION

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PREFACE

This book presents the standard topics of college algebra. I have tried to write so that students can learn as much as possible from the book itself. For that reason, the book contains a larger than usual number of examples, as well as explanations that are to the point but complete. Other features include the following.

Review Material The treatment of basic algebra in Chapter I is unusually complete. My own experience is that many students need and appreciate such review material, especially if it cannot be covered thoroughly in lectures. The detailed presentation in the text, along with the large numbers of examples and exercises, makes it easier for the instructor to concentrate on highlights and to move more quickly into the later parts of the book.

Exercises For a representative sample of the exercises that conclude each section, simply choose every third one beginning with either 1, 2, or 3. Each chapter concludes with a set of review exercises, and either the even- or the odd-numbered exercises will give a representative sample of these. Each exercise set begins with routine problems to check on the understanding of the basic ideas. Later exercises require more thought. Word problems are included at every good opportunity.

Answers An appendix gives answers to most of the odd-numbered exercises from each section, and to nearly all of the review exercises.

Flexibility Nearly all sections of the book can be covered in one lecture each. This implies that a one-semester course could start with Chapter I and continue through Chapter VIII. The book is flexible enough to allow for many other possibilities. For example, by spending only one week on Chapter I, and by omitting conic sections (in Chapter III), partial fractions (in Chapter V), and computation with logarithms (in Chapter VII), an instructor could begin a one-semester course with Chapter I, continue through Chapter VIII, and then cover one or more of Chapters IX, X, and XI.

Applications Applications are introduced at every reasonable opportunity. These generally occur at the end of a section so that they can be omitted without loss of continuity. Because of the importance of applications, and the interest they can add, instructors are encouraged to cover at least some of this material and students are urged to look at any portions that are omitted. The range of applications is broad.

Calculators Although algebra can be learned without a calculator, more interesting examples and exercises can be considered if a calculator is used for some of the computations. The symbol (C) marks the steps where a calculator has been used in examples; it also marks those exercises for which a calculator is suggested. In examples, the results of computations marked with (C) can be taken on faith. If a calculator is not available, then the exercises marked (C) can be either completed without a calculator or solved with the final computations only indicated and not carried out. Where the text gives advice on how to solve a problem with a calculator, the instructions will apply to many calculators but not to all of them; consult an owner's manual if you have questions.

Remarks on the Second Edition.

The major change from the first edition is in the first part of the book. The material that was in the first four chapters (18 sections) has been rewritten and is now covered in the first two chapters (11 sections). This change was made because the first edition was too slowly paced for many courses. Other changes are relatively minor but occur throughout the book; they mostly involve improvements made on the basis of experience and suggestions from students and other instructors. The material in this book is very similar to the algebra portion of my *College Algebra and Trigonometry* (John Wiley & Sons, 1984), except that this book contains more on matrices and determinants.

It is a pleasure to acknowledge the advice I have received from the following persons: James Schaedel (Western Illinois University); Rudolfo Maglio (Oakton Community College); and Charles Ziegenfus (James Madison University). It is also a pleasure to thank Carolyn Moore and the other people at John Wiley & Sons who helped with the production of this book.

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CHAPTER

I

BASIC ALGEBRA

This chapter introduces some of the basic ideas of algebra: real numbers, exponents, polynomials, and rational (fractional) expressions. The applications used as illustrations include simple and compound interest and examples from physical science. Section 1 is longer than other sections of the book in order to give a thorough review of the real numbers.

SECTION 1

REAL NUMBERS

A. Some Important Sets of Numbers

We begin with descriptions of the following sets of numbers, which are used throughout mathematics:

natural numbers
integers
rational numbers
irrational numbers
real numbers

Two other sets of numbers, imaginary and complex, will be described in Chapter VI.

The **natural numbers** are the numbers we count with:

$$1, 2, 3, 4, 5, \ldots$$

The **integers** are the natural numbers together with their negatives and zero:

$$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

The **positive integers** are the same as the natural numbers.

A number is **rational** if it can be written in the form a/b, where a and b are integers and $b \neq 0$. Examples are

$$\frac{2}{3}, \quad \frac{-43}{11}, \quad 0.5 = \frac{1}{2}, \quad 5\tfrac{1}{3} = \frac{16}{3}, \quad \text{and} \quad \frac{2\pi}{5\pi} = \frac{2}{5}.$$

Every integer is rational. For example,

$$-17 = \frac{-17}{1} \quad \text{and} \quad 0 = \frac{0}{1}.$$

(The word *rational* is used because of the connection between fractions and ratios. If, for instance, one object weighs $\frac{2}{3}$ as much as another, then their weights are in the *ratio* 2 to 3.)

Before defining real numbers and irrational numbers, it will be useful to look more closely at rational numbers. By division, we can represent any rational number as a decimal number. Here is the result for $\frac{2}{27}$.

$$\frac{2}{27} = 0.074074\cdots = 0.\overline{074}$$

To indicate that a block of digits is to be repeated, we write a bar over the block, as indicated above for 074. Here are some other examples.

$$3 = 3.0 \qquad \tfrac{1}{3} = 0.333\cdots = 0.\overline{3}$$

$$\tfrac{-7}{2} = -3.5 \qquad -\tfrac{103}{330} = -0.31212\cdots = -0.3\overline{12}$$

$$\tfrac{15}{8} = 1.875 \qquad \tfrac{16}{7} = 2.\overline{285714}$$

In the three examples on the left, the decimal representation *terminates:* there is a place after which only 0's would appear. In the three examples on the right, the decimal representation *repeats:* there is a place after which only the repetitions of some block that is not all 0's would appear. These two cases—terminating and repeating—are the only ones that can arise from rational numbers:

> A decimal number represents a rational number
> iff*
> it either terminates or repeats.

Appendix A explains why that statement is true. The statement gains interest because many decimal numbers neither terminate nor repeat. A simple example is 0.1010010001 · · ·, in which the number of 0's between the 1's increases by one each time.

Any number that has a decimal representation is called a **real number.** The real numbers that are not rational are called **irrational numbers.** Thus a real number is irrational iff it *cannot* be written as a quotient of two integers. Moreover, from what has already been said we can draw the following conclusion.

> A decimal number represents an irrational number
> iff
> it neither terminates nor repeats.

* We follow the practice, now widely accepted in mathematics, of using "iff" to denote "if and only if." This expression can be used only to connect statements that are equivalent. Here are some examples.

TRUE A number is a natural number *iff* it is a positive integer.
TRUE A number is a natural number *if* it is a positive integer.
TRUE A number is rational *if* it is an integer.
FALSE A number is rational *iff* it is an integer.

The example 0.1010010001 · · · is irrational. Figure 1.1 summarizes the relation between the sets of numbers discussed thus far; each set contains the sets that appear beneath it.

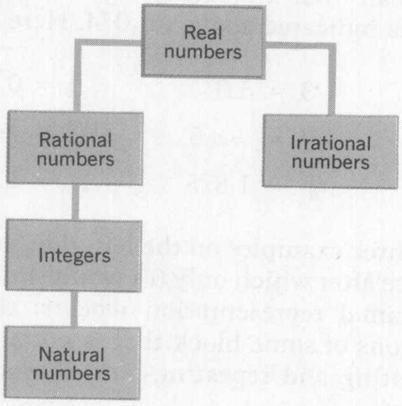

FIGURE 1.1

It can be proved that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$ are irrational.* In fact, if n is a positive integer, then $\sqrt{n}$ is irrational unless n is a perfect square (1, 4, 9, 16, 25, . . .). A proof that $\sqrt{2}$ is irrational is given in Appendix A. The number π, which represents the ratio of the circumference of any circle to its diameter, is also irrational (but the proof is not easy). These examples show that irrational numbers arise naturally, and not just from somewhat artificial constructions like 0.1010010001 · · ·.

Example 1.1 Which of the following are natural numbers? integers? rational numbers? irrational numbers?

$$8, \quad 1.414, \quad \frac{\pi}{2}, \quad \frac{\sqrt{25}}{3}, \quad 22.\overline{34}, \quad \sqrt{20} \quad -6\tfrac{2}{3}, \quad \frac{-85}{17}$$

Solution

Natural numbers: 8

Integers: 8 (Every natural number is an integer.)
$\frac{-85}{17}$ $\left(\frac{-85}{17} = -5\right)$

Rational numbers: 8, $\frac{-85}{17}$ (Every integer is a rational number.)
1.414 (Terminating decimal.)

* Square roots will be reviewed in Section 3. All that is needed here is that $\sqrt{2}\sqrt{2} = 2$, $\sqrt{3}\sqrt{3} = 3$, and so on.

$\sqrt{25}/3$ ($\sqrt{25}/3 = \frac{5}{3}$)

$22.\overline{34}$ (Repeating decimal.)

$-6\frac{2}{3}$ ($-6\frac{2}{3} = \frac{-20}{3}$)

Irrational numbers: $\pi/2$ (If $\pi/2$ were a fraction of integers, say $\pi/2 = a/b$, then π would also be a fraction of integers, $\pi = 2a/b$.)

$\sqrt{20}$ (20 is not a perfect square.)

Because $\sqrt{2}$ and π are irrational, no calculator can represent them exactly. A calculator with eight-digit accuracy will use the rounded-off values 1.4142136 for $\sqrt{2}$ and 3.1415927 for π. Such approximations are perfectly adequate for most practical work, but even more accurate values like 1.41421356237309504880 and 3.14159265358979323846 are not exact, since a terminating decimal cannot be exactly equal to an irrational number. Whenever the word *number* is used without qualification in this book, it will mean *real number*.

B. Real Lines

Much of the importance of the real numbers derives from their use for measurement. This, in turn, is based on the assumption of a one-to-one correspondence between the set of real numbers and the set of points on a line. This correspondence can be described as follows. Choose two points on a straight line and label them 0 and 1 (Figure 1.2). Make the line into a *directed line* by taking the direction from 0 toward 1 to be positive. The integers correspond to the points on this line in the obvious way shown in the figure. The other rational numbers arise when the segments between integer points are divided into equal parts (halves, thirds, fourths, and so on). For example, $\frac{1}{2}$ corresponds to the point midway between 0 and 1. In this way the rational numbers will account for a vast number of the points on the line. Our remarks show, however, that there will still be points—such as those corresponding to $\sqrt{2}$ and π—that will not be accounted for by rational numbers. Again, the basic assumption about measurement and numbers is that there is precisely one real number for each point and precisely one point for each real number. A line together with such a correspondence is called a **real line** (or **number line** or **coordinate line**). The number associated with a point on a real line is called the **coordinate** of the point. Figure 1.3 shows examples of points with rational coordinates.

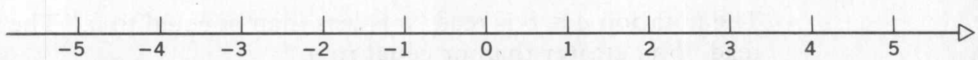

FIGURE 1.2

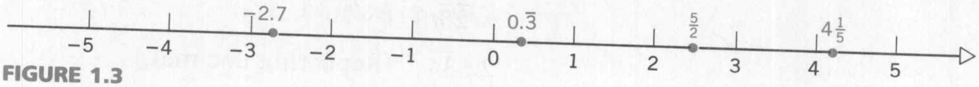

FIGURE 1.3

Points with irrational coordinates are located using decimal or fractional approximations. Figure 1.4 shows several examples. Figure 1.5 shows a magnified piece of Figure 1.4 with several additional points labeled to indicate the position of π; notice that $\frac{22}{7}$, which is often used as an approximation for π, is slightly larger than π. Usually, 3.14 will suffice as an approximate decimal coordinate for π.

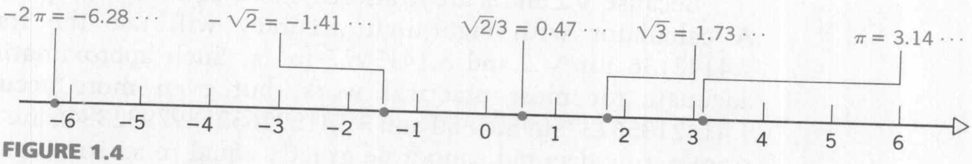

FIGURE 1.4

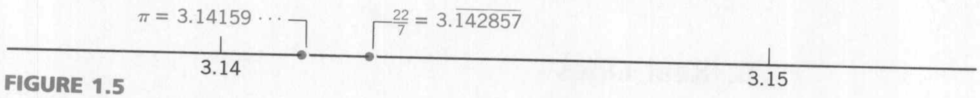

FIGURE 1.5

Instead of saying "the point with coordinate c" when referring to a point on a real line, we generally just say "the point c" if there is no chance of confusion.

Consider a horizontal real line directed to the right. If a and b are real numbers, and a is to the *left* of b on the line, then we say that a is **less than** b, and we write $a < b$. Thus

$$3 < 6, \quad -17 < 2, \quad -20 < -17, \quad \text{and} \quad 0 < \pi.$$

The notation $b > a$ means the same as $a < b$. Thus

$$5 > 2, \quad 5 > -2, \quad 2 > -5, \quad \text{and} \quad -2 > -5.$$

We say that b is **greater than** a if $b > a$.

The notation $a \le b$ means that either $a = b$ or $a < b$. Similarly for $b \ge a$. Thus

$$4 \le 9, \quad 9 \le 9, \quad -1 \ge -3, \quad \text{and} \quad -1 \ge -1.$$

The notation $a \le b$ is read "a is less than or equal to b." The notation $b \ge a$ is read "b is greater than or equal to a."

A number c is **positive** if $c > 0$, and **negative** if $c < 0$. Statements involving

$<, >, \leq$, or $\geq$ are called **inequalities.** More will be said about inequalities in Section 10.

C. Properties of the Real Numbers

This subsection summarizes most of the basic properties of the real numbers. The first few of these properties are *axioms* (statements that are assumed to be true); the other properties are *theorems* (statements that can be proved from the axioms). Although the theorems will be distinguished from the axioms, the proofs of the theorems will be omitted. The important point for our purposes is to be able to use the properties correctly. In all of the properties the letters a, b, and c denote real numbers.

Commutative properties

$a + b = b + a$ for all a and b.

$ab = ba$ for all a and b.

Example for addition

$$3 + 5 = 8 \quad \text{and} \quad 5 + 3 = 8$$

Associative properties

$a + (b + c) = (a + b) + c$ for all a, b, and c.

$a(bc) = (ab)c$ for all a, b, and c.

Example for multiplication

$$3 \cdot (8 \cdot 4) = 3 \cdot 32 = 96 \quad \text{and} \quad (3 \cdot 8) \cdot 4 = 24 \cdot 4 = 96$$

Distributive properties

$a(b + c) = ab + ac$ for all a, b, and c.

$(a + b)c = ac + bc$ for all a, b, and c.

Example of the first distributive property

$$3(2 + 4) = 3 \cdot 6 = 18 \quad \text{and} \quad 3 \cdot 2 + 3 \cdot 4 = 6 + 12 = 18$$

Zero and one

$a + 0 = 0 + a = a$ for every a.

$a \cdot 1 = 1 \cdot a = a$ for every a.

Negatives. For each number a there is a unique number denoted $-a$, and called the **negative** (or **additive inverse**) of a, such that

$$a + (-a) = (-a) + a = 0.$$

Reciprocals. For each number a except 0 there is a unique number denoted $1/a$, and called the **reciprocal** (or **multiplicative inverse**) of a, such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1.$$

The preceding properties in this section are called the **field axioms** (because any system that satisfies them, like the real numbers, is called a **field**). We also need the following properties of equality.

Equality

$a = a$ for every a. *Reflexive property*

If $a = b$, then $b = a$. *Symmetric property*

If $a = b$ and $b = c$, then $a = c$. *Transitive property*

If $a = b$ and $c = d$, then
$a + c = b + d$ and $ac = bd$. *Substitution property*

The remaining properties in this section can be proved from the field axioms and the preceding properties of equality.

Other properties of equality

If $a = b$, then $a + c = b + c$ and $ac = bc$ for every c.

If $a + c = b + c$, then $a = b$. *Cancellation for addition*

If $ac = bc$ and $c \neq 0$, then $a = b$. *Cancellation for multiplication*

Properties of zero

$a \cdot 0 = 0 \cdot a = 0$ for every a.

$ab = 0$ iff either $a = 0$ or $b = 0$.

Examples

$$0 + 8 = 8, \quad \text{but} \quad 0 \cdot 8 = 0.$$

$$\text{If} \quad 3x = 0, \quad \text{then} \quad x = 0.$$

Subtraction. This property is defined in terms of addition and negatives, as follows:

$$a - b = a + (-b) \qquad \text{for all } a \text{ and } b.$$

Example

$$4 - 9 = 4 + (-9) = -5$$

Other properties of negatives. For all real numbers a, b, and c,

$$-(-a) = a$$
$$-(a + b) = (-a) + (-b)$$
$$(-a)b = a(-b) = -(ab)$$
$$(-a)(-b) = ab$$
$$a(b - c) = ab - ac.$$

Examples

$$-(-7) = 7$$
$$-(3 + 8) = (-3) + (-8) = -11$$
$$(-2)3 = 2(-3) = -(2 \cdot 3) = -6$$
$$(-3)(-2) = 3 \cdot 2 = 6$$
$$5(8 - 2) = 5 \cdot 6 = 30 \quad \text{and} \quad 5 \cdot 8 - 5 \cdot 2 = 40 - 10 = 30$$