Constants in Some Inequalities of Analysis

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Preface

Inequalities play an important role in various questions of Calculus and Functional Analysis; they connect certain quantities which are significant for the problem under consideration. We think of such inequalities like estimates for the norm of an operator, error estimates in numerical methods, and estimates for the norm of a function that is extended to some larger domain; furthermore inequalities characterizing the accuracy in approximating a function etc.

Mainly those inequalities contain certain "constant" factors which depend on some quantities of the problem considered. The values of the constants are usually not specified. Therefore, two problems arise in that connection; the first is to determine the *best* constant that assures the inequality to hold. Such constants are also called *sharp* or *exact* ones. The other problem consists of evaluating effectively some numerical value of the constant for which the inequality considered is true. That value should, obviously, be as near as possible to the best one. There are some inequalities for which one of those problems or even both of them can be solved. A large variety of such examples is given in [8] and [2]. We quote some of them here; they are taken from the Appendix to the Russian translation of the book [8].

1. Karlson's inequality. Let a_n be non-negative numbers which do not vanish all together, then

$$(\sum a_n)^4 < \pi^2 \sum a_n^2 \sum n^2 a_n^2;$$

the constant π^2 cannot be made smaller, although the inequality itself may be strengthened:

$$(\sum a_n)^4 < \pi^2 \sum a_n^2 \sum (n - \tfrac{1}{2})^2 a_n^2.$$

2. There is a continuous analogue of Karlson's inequality. If $f(x) \geqq 0$ and $f(x) \not\equiv 0$ then

$$\left(\int_0^\infty f(x)\,\mathrm{d}x\right)^4 < \pi^2 \int_0^\infty f^2(x)\,\mathrm{d}x \int_0^\infty x^2 f^2(x)\,\mathrm{d}x.$$

3. Let the function $u(x)$ be 2π-periodic, $\int_0^{2\pi} u(x)\,\mathrm{d}x = 0$, and the k-th derivative $u^{(k)}$ belong to $L_s(0, 2\pi)$. If $1 \leqq s \leqq r \leqq \infty$ and $\frac{1}{\mu} := 1 + \frac{1}{r} - \frac{1}{s}$

then

$$\|u\|_{L_r(0,2\pi)} \leq C_{k\mu} \|u^{(k)}\|_{L_\mu(0,2\pi)}$$

where $C_{k\mu} := \min_{\xi} \|\varphi_k - \xi\|_{L_\mu(0,2\pi)}$ with $\varphi_k(t) := \frac{1}{\pi} \sum_{n=1}^{\infty} n^{-k} \cos\left(nt - \frac{k\pi}{2}\right)$.

Let us consider another example, cf. [26], Chap. III, § 13. Let the function $u(x)$ be 2π-periodic and have a bounded k-th derivative, $|u^{(k)}(x)| \leq M$. If $E_n(u)$ denotes the best approximation (in the C-metric) of the function u by trigonometric polynomials of order n then

$$E_n(u) \leq A_k M n^{-k}, \qquad E_n(u) \leq B_k \omega\left(u^{(k)}, \frac{2\pi}{n}\right) n^{-k}.$$

Here ω is the modulus of continuity, $B_k := A_k + (2\pi)^{-1} A_{k+1}$, and A_k can be evaluated by

$$A_k := \frac{2^{k+2}}{\pi} \int_0^{\infty} |H_k(t)| \, dt$$

where $H_0(t) := t^{-2}(\cos t - \cos 2t)$, $H_i(t) := \int_t^{\infty} H_{i-1}(\tau) \, d\tau$.

There is a considerable number of such examples, where the constants, sometimes even the best ones, can be evaluated. However, for many important problems the determination of the corresponding constants has encountered reasonable difficulties. We mention here a relatively simple example.

Let Ω be a bounded domain in the m-dimensional Euclidean space $\mathbb{E}^m$. As usual, $\mathring{W}_2^1(\Omega)$ denotes the subspace of functions belonging to the Sobolev space $W_2^1(\Omega)$ which vanish at the boundary $\partial\Omega$ of the domain Ω. These functions satisfy the so-called Friedrichs inequality,

$$\int_\Omega |u(x)|^2 \, dx \leq \varkappa \int_\Omega |\nabla u|^2 \, dx, \quad \varkappa = \text{const}, \tag{1}$$

sometimes also connected with the name of Poincaré. Here the factor $\varkappa$ depending on Ω occurs. Clearly the value of $\varkappa$ for which (1) holds is not unique: If the inequality is true for some $\varkappa'$ then it is true for any value $\varkappa > \varkappa'$ also. Hence, the best constant is the smallest value of the $\varkappa$-s; we shall denote it by $\varkappa_0$. It is not difficult to characterize $\varkappa_0$, namely, it is the reciprocal to the smallest eigenvalue of the problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2}$$

In that case it is also not difficult to indicate effectively a value $\varkappa$ for which (1) is true. We enclose Ω in a rectangular parallelepiped with the edges $a_1, a_2, \ldots, a_m$. The smallest eigenvalue of the problem (2) for that parallelepiped is $\pi^2(a_1^{-2} + a_2^{-2} + \ldots + a_m^{-2})$. An expansion of the domain does not make the smallest eigenvalue of the problem (2) increase, therefore $\varkappa_0 \leq \pi^{-2}(a_1^{-2} + a_2^{-2} + \ldots + a_m^{-2})^{-1}$. Any number not smaller than $\varkappa_0$ satisfies the inequality (1) such that we may put

$$\varkappa := \pi^{-2}(a_1^{-2} + a_2^{-2} + \ldots + a_m^{-2})^{-1}. \tag{3}$$

The smaller the parallelepiped enclosing Ω is the closer the value (3) is to $\varkappa_0$.

Let us consider another example. The inequality (1) is also true for those functions of $W_2^1(\Omega)$ which have a zero mean value over Ω (in that case it is called Poincaré inequality) provided that the domain Ω satisfies the so-called cone condition. Here, the first problem mentioned above can be solved in the theoretical plane simply: The smallest $\varkappa$ is equal to the reciprocal of the smallest positive eigenvalue of the Neumann problem for the domain Ω. The second problem is much more difficult: the technique for obtaining simple and effective lower estimates for the eigenvalues of the Neumann problem is still unsufficiently developed.

From that example one can already imagine the considerable effort which may be required for evaluating effectively a concrete constant. To determine the best constant numerically can be particularly difficult. It is not by chance that the corresponding constants in some inequalities are not specified. By the way, exceptions from that "unfair" rule do exist. We mention here, in particular, many important contributions to the Constructive Function Theory, which are devoted to indicating the best values for the constants in inequalities characterizing the rate of approximation. However, in general one is frequently satisfied with proving the existence of the constants needed. In many cases such an argument is sufficient; but sometimes an information about the numerical value of the constants is quite desirable. To give an idea of what we have in mind, let us consider the following important example.

In various finite difference and finite element methods the error is estimated by a term $C\varphi(h)$ where h denotes the mesh parameter, and φ is an increasing function of h with $\varphi(0) = 0$; mainly $\varphi(h)$ is a power of h. The constant C is frequently not specified, and therefore the practical meaning of the estimate is poor: we know only that the error estimate becomes smaller if the parameter h is decreasing, but we cannot say whether the estimate is small or large for concrete values of h. Similar arguments can be found in the book [17].

The present booklet is essentially based on the author's results and contains five chapters. The first two chapters are devoted to extension constants; so we shall call the factor in norm estimates for an extended function. In various investigations we experienced the significance of the extension constants; they play an important role in estimates for the constants in other inequalities of Analysis. In Chap. I we derive estimates for the constants of the extensions suggested by H. Whitney, R. M. Hestenes, and A. P. Calderon. Here we develop also an extension procedure proposed by the author that yields the smallest extension constant for functions of Sobolev classes.

In Chap. II we determine the exact value of the smallest extension constant for functions of the class W_2^1 originally defined on a ball. Then we are able, in particular, to estimate that constant for star-like domains with a boundary of certain smoothness. Furthermore, we evaluate the smallest constant for the extension from the exterior to the interior of a ball.

In Chap. III we consider two problems in the theory of Sobolev spaces, namely a theorem on equivalent norms and a theorem about the mollification of a function. Both problems are connected with certain inequalities, and the corresponding constants are estimated, too.

Chap. IV is devoted to the constants in error estimates of finite element approximations over a cubic grid (cf. the author's book [11]). The basic functions are taken as piecewise polynomials, and they lead to the best rate of approximation with respect to the mesh parameter h. In that chapter we widely use the results of the foregoing chapters.

Chap. V was written by S. V. POBORCHI; it is based on his own results. The same problems as in chapters III and IV are considered, only with respect to the so-called anisotropic Sobolev spaces. I take the opportunity to express my deep gratitude to S. V. POBORCHI for compiling this chapter.

Throughout the book, except in the second chapter, we do not aim at indicating the best values for the constants. We take the problem as solved if we succeed in evaluating a concrete value for the constant that makes the inequality considered true.

Leningrad, March 1980 S. G. MIKHLIN

I was pleased to learn that the present booklet published in German a few years ago should be translated into English. Let me take the opportunity to express my very deep gratitude to the editors of the TEUBNER-TEXTE zur Mathematik for including it into the series, and for promoting the project of an English version. Finally, I very sincerely thank DR. REINHARD LEHMANN, Halle, who made the German translation as well as the English one.

Leningrad, March 1984 S. G. MIKHLIN

I. The extension constant

§ 1. The problem

1.1. Let Ω and Ω' be two sets of the m-dimensional Euclidean space $\mathbb{E}^m$ with $\Omega \subset \Omega'$. A function u_1 defined on Ω' is called an extension to Ω' of a given function u defined on Ω if

$$u_1(x) = u(x), \quad x \in \Omega. \tag{1.1}$$

The problem of extending a function can be stated in a sufficiently general manner as follows.

Let $\mathscr{B}$ and $\mathscr{B}_1$ be two Banach spaces of functions defined on Ω and Ω' resp. To every function $u \in \mathscr{B}$ find its extension $u_1 \in \mathscr{B}_1$ such that

$$\|u_1\|_{\mathscr{B}_1} \leqq C\|u\|_{\mathscr{B}} \tag{1.2}$$

with a constant C that does not depend on u.

If Ω is bounded and Ω_1 unbounded one adds usually the condition that the extended function u_1 vanishes outside of a certain ball, one and the same for all $u \in \mathscr{B}$.

The smallest value of C for which the inequality (1.2) is true, is called *extension constant*. It depends, in general, on the spaces $\mathscr{B}$ and $\mathscr{B}_1$, and also on the procedure which relates the extension u_1 to the function u. If that procedure results in a unique extension, we may speak of an extension operator. In that case the extension constant is just the norm of the extension operator.

1.2. Frequently the Sobolev spaces $W_p^s(\Omega)$ and $W_p^s(\Omega')$ ($p \geqq 1$, s integer) resp., or the spaces $C^s(\bar{\Omega})$ and $C^s(\bar{\Omega}')$ ($s \geqq 0$) resp., are taken as the spaces $\mathscr{B}$ and $\mathscr{B}_1$. In that case we speak of "an extension of functions of the class $W_p^s(\Omega)$ (or $C^s(\bar{\Omega})$) to the set Ω' preserving the class". The class preserving extension to the whole space $\mathbb{E}^m$ is of particular interest.

One of the first contributions devoted to the extension of functions was given by H. Whitney [24], where an extension of functions of the class $C^s(F)$

(F being a closed subset of $\mathbb{E}^m$) to the whole space preserving the class is considered. Another procedure for extending functions of the class $C^s(\bar{\Omega})$ (Ω being a bounded domain and s integer) to the whole space $\mathbb{E}^m$ preserving the class was suggested by R. M. HESTENES [9]. V. M. BABICH [1] and S. M. NIKOLSKI [16] applied the procedure of HESTENES to the class $W_p^s(\Omega)$. A. P. CALDERON [4] developed a procedure for the extension of functions of $W_p^s(\Omega)$ to the whole space which is based on the integral representation formula for such functions. The papers of H. WHITNEY and A. P. CALDERON are briefly exposed in the book of E. M. STEIN [22]; there can be found another procedure for the extension of functions of $W_p^s(\Omega)$ to $\mathbb{E}^m$ preserving the class. It is worth noting that STEIN's procedure leads to an extension operator that does not depend on s, p, and Ω. A proof of CALDERON's theorem is also given in the book [11].

In [11, 14] we proposed an extension procedure for functions of the class $W_p^s(\Omega)$ which leads to the smallest extension constant.

In the subsequent sections of chapters I and II we consider, more or less completely, estimates for the extension constants of the extension procedures mentioned above (except STEIN's procedure); the procedures themselves are briefly exposed, too.

§ 2. The Whitney extension

We are going to consider here the simplest case where $\mathscr{B} := C(F)$ and $\mathscr{B}_1 := C(\mathbb{E}^m)$ with a bounded closed set F of $\mathbb{E}^m$.

Let $u(x)$ be an arbitrary function continuous on F. Its Whitney extension to $\mathbb{E}^m$ is constructed as follows, cf. [22]. By F_C we denote the complement $\mathbb{E}^m \setminus F$ of F in $\mathbb{E}^m$. We give a sequence of cubes Q_k with the properties

(i) the edges of the cubes are parallel to the coordinate axes;
(ii) for $i \neq k$ the interiors of the cubes Q_k and Q_i do not intersect;
(iii) $\operatorname{diam} Q_k \leqq \operatorname{dist}(Q_k, F) \leqq 4 \cdot \operatorname{diam} Q_k$;
(iv) $\bigcup_{k=1}^{\infty} Q_k = F_C$.

For every natural number k there is a point p^k of F such that $\operatorname{dist}(Q_k, F) = \operatorname{dist}(Q_k, p^k)$. Furthermore, we fix in F_C a partition of unity as follows. Choose a function $\varphi(x)$ with the properties

$$\varphi \in C^\infty(\mathbb{E}^m), \quad 0 \leqq \varphi(x) \leqq 1;$$
$$\varphi(x) \equiv 1 \quad \text{in the cube } -1 \leqq x_i \leqq 1 \quad (1 \leqq i \leqq m);$$
$$\varphi(x) = 0 \quad \text{outside of the cube } -1-\varepsilon < x_i < 1+\varepsilon \quad (1 \leqq i \leqq m);$$

with a fixed positive real ε (x_i denotes the i-th cartesian coordinate of the point x). Let x^k denote the center of the cube Q_k and l_k the edge length. Then we put $\varphi_k(x) := \varphi((x - x^k)/l_k)$ such that, obviously, $\varphi_k(x) = 1$, $x \in Q_k$, and $\varphi_k(x) = 0$, $x \bar{\in} Q_k^*$, where Q_k^* is the cube Q_k homothetically enlarged by the coefficient $1+\varepsilon$. Finally, we put

$$\varphi_k^*(x) := \frac{\varphi_k(x)}{\Phi(x)}, \qquad \Phi(x) := \sum_{k=1}^{\infty} \varphi_k(x), \quad x \in F_C.$$

It is clear that $0 \leqq \varphi_k^*(x) \leqq 1$ and $\sum_{k=1}^{\infty} \varphi_k^*(x) = 1$.

The extension of the function $u(x)$ to $\mathbb{E}^m$ is given by

$$u_1(x) := \begin{cases} u(x), & x \in F; \\ \sum_{k=1}^{\infty} u(p^k)\, \varphi_k^*(x), & x \in F_C; \end{cases} \tag{2.1}$$

and $u_1 \in C(\mathbb{E}^m)$ can be proved.

In that case it is not difficult to determine the extension constant. The first of the equations (2.1) yields $\|u_1\|_{C(\mathbb{E}^m)} \geqq \|u\|_{C(F)}$, and the second one is estimated

$$\forall x \in F_C, |u_1(x)| \leqq \max_k |u(p^k)| \sum_{k=1}^{\infty} \varphi_k^*(x) = \max_k |u(p^k)| \leqq \|u\|_{C(F)}.$$

On the other hand, for $\forall x \in F$ we have $|u_1(x)| \leqq \|u\|_{C(F)}$ such that $\|u_1\|_{C(\mathbb{E}^m)} \leqq \|u\|_{C(F)}$ and therefore $\|u_1\|_{C(\mathbb{E}^m)} = \|u\|_{C(F)}$. The extension constant is equal to 1.

The extension u_1 may be required to vanish outside of a certain ball, and it is natural to assume F contained in the interior of that ball. Let R denote the radius of the ball, and δ be an arbitrary positive number. To make the function u_1 zero outside of the ball it is sufficient to multiply it by a "cutting" function, i.e. by a continuous function which is 1 if $|x| \leqq R$, and 0 if $|x| > (1+\delta) R$, its range varying between 0 and 1 if $R < |x| < (1+\delta) R$. The extension constant remains still equal to 1.

§ 3. The Hestenes extension

3.1. Here we want to extend a function $u \in C^s(\bar{\Omega})$ from the domain $\Omega \subset \mathbb{E}^m$ to a larger one. To begin with, we assume Ω to be contained in the half-space $x_m > 0$, where a part Γ' of the boundary $\Gamma = \partial\Omega$ forms a certain $(m-1)$-dimensional closed region of the plane $x_m = 0$. Following HESTENES, a function u can be extended preserving the class to the domain Ω' situated symmetrically to Ω with respect to the plane $x_m = 0$: Let $x = (x', x_m)$ and put

$$\forall x \in \Omega', \ u_1(x) = u_1(x', x_m) := \sum_{k=0}^{s} \lambda_k u(x', -\varepsilon_k x_m), \tag{3.1}$$

where $0 < \varepsilon_k < 1$, and ε_k decreases with k increasing; the coefficients λ_k satisfy the simultaneous equations

$$\sum_{k=0}^{s} (-\varepsilon_k)^j \lambda_k = 1, \quad j = 0, 1, \ldots, s. \tag{3.2}$$

Now we are going to estimate the constant for the extension of $u \in C^s(\bar\Omega)$ from Ω to $\Omega_1 := \Omega \cup \Omega'$. To that aim we shall estimate $\|u_1\|_{C^s(\bar\Omega')}$, where the norm in $C^s(\bar\Omega)$ is defined as

$$\|u\|_{C^s(\bar\Omega)} := \|u\|_{C(\bar\Omega)} + \sum_{|\alpha|=s} \|u^{(\alpha)}\|_{C(\bar\Omega)}, \tag{3.3}$$

and similarly for other domains. From (3.1) we find

$$\|u_1\|_{C^s(\bar\Omega')} \leqq \|u\|_{C^s(\bar\Omega)} \sum_{k=0}^{s} |\lambda_k|. \tag{3.4}$$

The numbers λ_k can be explicitly expressed by the ε_j: Indeed, λ_k is a quotient with the denominator being the Vandermonde determinant of the numbers $-\varepsilon_0, -\varepsilon_1, \ldots, -\varepsilon_s$, and the numerator being the same determinant only with 1 in place of $-\varepsilon_k$. Therefore,

$$|\lambda_k| = \prod_{j=0}^{s}{}' (1 + \varepsilon_j) \Big/ \prod_{j=0}^{s}{}' |-\varepsilon_k + \varepsilon_j|, \tag{3.5}$$

where the prime indicates omitting $j = k$. Now it is clear that the extension constant in the case considered can be estimated from above by the quantity

$$\sum_{k=0}^{s} \prod_{j=0}^{s}{}' (1 + \varepsilon_j) \Big/ \prod_{j=0}^{s}{}' |-\varepsilon_k + \varepsilon_j|. \tag{3.6}$$

3.2. For special choices of the numbers ε_k we can derive from (3.6) more specific estimates. Let us, for example, put $\varepsilon_k = 1/(k+1)$, then we obtain for the numerator in (3.5) a simple estimate,

$$\prod_{=0}^{s}{}' \left(1 + \frac{1}{j+1}\right) < \prod_{j=0}^{s} \left(1 + \frac{1}{j+1}\right)$$

$$= \prod_{j=1}^{s+1} \left(1 + \frac{1}{j}\right) < \exp\left(\sum_{j=1}^{s+1} \frac{1}{j}\right) < \exp\left(\ln(s+1) + \bar c\right) = e^{\bar c}(s+1)$$

where $\bar c = 0.57721566\ldots$ is the Euler constant. For the denominator in (3.5),

$$\prod_{j=0}^{k-1} \left(\frac{1}{j+1} - \frac{1}{k+1}\right) \prod_{j=k+1}^{s} \left(\frac{1}{k+1} - \frac{1}{j+1}\right),$$

we have

$$\prod_{j=0}^{k-1} \left(\frac{1}{j+1} - \frac{1}{k+1}\right) = \prod_{j=0}^{k-1} \frac{k-j}{(j+1)(k+1)} = \frac{1}{(k+1)^k}$$

and

$$\prod_{j=k+1}^{s} \left(\frac{1}{k+1} - \frac{1}{j+1}\right) = \prod_{j=k+1}^{s} \frac{j-k}{(j+1)(k+1)} = \frac{(k+1)!\,(s-k)!}{(k+1)^{s-k}(s+1)!}$$

such that

$$|\lambda_k| \leqq e^{\bar c}(s+1)(k+1)^s \binom{s+1}{k+1} < e^{\bar c} 2^s (s+1)(k+1)^s.$$

Hence, we arrive at

$$\sum_{k=0}^{s} |\lambda_k| < e^{\bar c} 2^s (s+1) \sum_{k=0}^{s} (k+1)^s < e^{\bar c} 2^s (s+1) \int_1^{s+2} k^s \, dk = e^{\bar c} 2^s [(s+2)^{s+1} - 1],$$

and the extension constant is estimated from above by

$$e^{\bar{c}}\, 2^s[(s+2)^{s+1} - 1] = 1.781 \cdot 2^s[(s+2)^{s+1} - 1]. \tag{3.7}$$

3.3. Now let us consider the extension through a part Γ_0 of the boundary $\partial\Omega$, where we assume Γ_0 to be given in an appropriate local coordinate system by the equation

$$x_m = f(x'), \quad f \in C^s. \tag{3.8}$$

The mapping

$$y_j = x_j, \quad 1 \leqq j \leqq m-1, \qquad y_m = x_m - f(x') \tag{3.9}$$

transforms a part Ω_0 of the given domain Ω into a certain domain Ω'_0, and Γ_0 is transformed into a part of the plane $y_m = 0$. By $u'(y)$ we denote the function $u(x)$ (more precisely, its restriction to Ω_0) under the mapping (3.9), and by $u'_1(y)$ the Hestenes extension of $u'(y)$ through the part of the boundary $y_m = 0$, where the numbers ε_k are chosen as indicated in 3.2. Obviously, there are two positive constants C_1 and C_2 such that for any function $u \in C^s(\bar{\Omega})$ —

$$C_1 \|u\|_{C^s(\bar{\Omega}_0)} \leqq \|u'\|_{C^s(\bar{\Omega}_0')} \leqq C_2 \|u\|_{C^s(\bar{\Omega}_0)}. \tag{3.10}$$

Those constants depend on the function f, and it is not difficult to determine them; we may assume them to be given.

By Ω'^* we denote that domain into which the function $u'(y)$ is extended, and by Ω^* the image of the domain Ω'^* under the inverse mapping of (3.9) $x_j = y_j$, $1 \leqq j \leqq m-1$, $x_m = y_m + f(y')$. Finally, let $\Omega_1 := \Omega \cup \Omega^*$. We may assume the constants C_1 and C_2 to satisfy an inequality (3.10) for the domains Ω^* and Ω'^*, too. Thus, we arrive at

$$\|u_1\|_{C^s(\overline{\Omega_0 \cup \Omega^*})} \leqq \frac{C_2}{C_1} e^{\bar{c}}\, 2^s[(s+2)^{s+1} - 1]\, \|u\|_{C^s(\bar{\Omega}_0)}, \tag{3.11}$$

from which a corresponding inequality with Ω instead of Ω_0 and $\Omega \cup \Omega^*$ instead of $\Omega_0 \cup \Omega^*$ is easily derived. Now it is clear that the extension constant in our case can be estimated from above by

$$\frac{C_2}{C_1} e^{\bar{c}}\, 2^s[(s+2)^{s+1} - 1]. \tag{3.12}$$

3.4. Let Ω be an arbitrary domain belonging to the class C^s. We shall construct a locally finite covering of $\mathbb{E}^m$ in the following way. Every point of the boundary $\partial\Omega$ is taken as the center of a ball B with a sufficiently small radius such that the part of $\partial\Omega$ contained in the ball admits to a representation (3.9) in a local coordinate system. From the family of those balls we select by the Heine-Borel Theorem a finite number of them, say $B_1, \ldots, B_N$; they form a covering of a certain two-sided vicinity of $\partial\Omega$. Let B_0 be a domain of the class C^s such that $B_0 \subset \Omega$ and $B_0, B_1, \ldots, B_N$ form a covering of Ω. Finally, take balls $B_{N+1}, B_{N+2}, \ldots$ such that they do not intersect with Ω, and $B_0, B_1, \ldots, B_N, B_{N+1}, \ldots$ form a locally finite covering of $\mathbb{E}^m$.

Let $\{\varphi_n(x)\}$ be a partition of unity subordinate to the covering $\{B_n\}$. Obviously, $\varphi_n(x) = 0$ if $n > N$ and $x \in \Omega$. Then we have

$$\forall x \in \Omega, \quad u(x) = \sum_{k=0}^{N} u(x)\,\varphi_k(x).$$

Each function $u_k(x) := u(x)\,\varphi_k(x)$ is extended in the following way: For $k = 0$ put $u_{k1}(x) = 0$, $x \bar{\in} \Omega$; for $1 \leqq k \leqq N$ take u_{k1} as constructed in section 3.3 multiplied by an appropriate cut-off function. As a result we obtain the extension

$$u_1(x) := \sum_{k=0}^{N} u_{k1}(x),$$

and by virtue of the considerations above it is obvious that

$$\|u_1\|_{C^s(\mathbb{E}^m)} \leqq \gamma_s\, e^{\bar{c}}\, 2^s[(s+2)^{s+1} - 1]\, \|u\|_{C^s(\Omega)}, \tag{3.13}$$

where γ_s is a constant depending on the geometric properties of the domain Ω. Hence, we obtain in the case considered the following estimate from above for the extension constant

$$\varkappa(s, \Omega) \leqq \gamma_s\, e^{\bar{c}}\, 2^s[(s+2)^{s+1} - 1]. \tag{3.14}$$

As one can see from (3.14), the estimate for the extension constant consists of two factors, one depending on the domain Ω and the other on the extension procedure.

3.5. Using similar arguments, we may estimate the constant of the Hestenes extension for functions belonging to Sobolev spaces, where we have to impose additional restrictions on Ω. It is worth noting that the estimate for the extension constant obtained may be improved if for the numbers ε_k one takes, e.g. $\varepsilon_k = -(k+1)^{-1/2}$ instead of $\varepsilon_k = -(k+1)^{-1}$. We will not dwell on hese questions any longer; they are exposed in the paper [13] of the author.

§ 4. The Calderon extension

4.1. We shall say that a surface $\Gamma \subset \mathbb{E}^m$ is *Lipschitz-continuous* and write $\Gamma \in C^{(0,1)}$ if Γ has the following characterization: For any point $x \in \Gamma$ there is a local cartesian coordinate system $(\xi_1, \ldots, \xi_m)$ with origin at the point x, such that the part of the surface Γ contained in some neighborhood of x has the representation

$$\xi_m = f(\xi'), \quad \xi' := (\xi_1, \ldots, \xi_{m-1}), \tag{4.1}$$

and the function f satisfies a Lipschitz condition

$$|f(\xi') - f(\xi'')| \leqq A|\xi' - \xi''| \tag{4.2}$$

with a constant A not depending on x. The class of bounded domains with the boundary being a Lipschitz-continuous surface coincides with the class of domains satisfying the cone condition or, equivalently, with the class of domains which can be represented as the union of a finite number of domains