STUDENT SOLUTIONS MANUAL

SALAS AND HILLE'S CALCULUS

SEVERAL VARIABLES CHAPTERS 12–17

SEVENTH EDITION

REVISED BY
GARRET J. ETGEN

STUDENT SOLUTIONS MANUAL

to Accompany

SALAS AND HILLE'S

CALCULUS

SEVERAL VARIABLES

Chapters 12-17

Seventh Edition

REVISED BY

GARRET J. ETGEN

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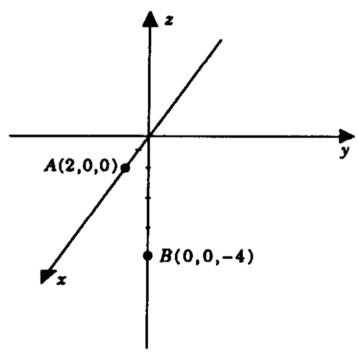
CHAPTER 12

SECTION 12.1

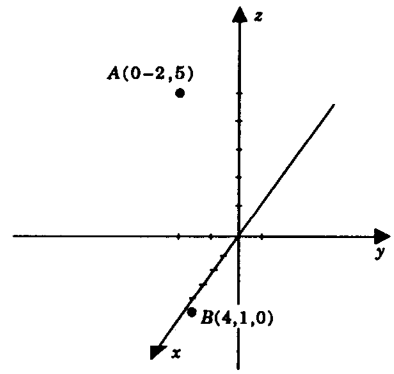
1.



length $\overline{AB}$: $2\sqrt{5}$

midpoint: $(1, 0, -2)$

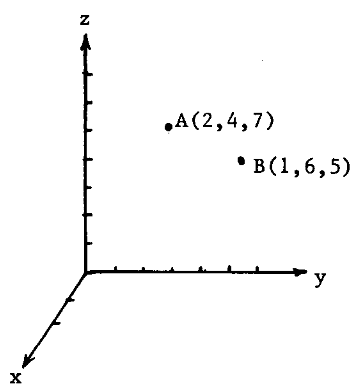
3.



length $\overline{AB}$: $5\sqrt{2}$

midpoint: $(2, -\frac{1}{2}, \frac{5}{2})$

5.



length $\overline{AB}$: 3 midpoint: $(\frac{3}{2}, 5, 6)$

7. $z = -2$ **9.** $y = 1$ **11.** $x = 3$

13. $x^2 + (y-2)^2 + (z+1)^2 = 9$ **15.** $(x-2)^2 + (y-4)^2 + (z+4)^2 = 36$

17. $(x-3)^2 + (y-2)^2 + (z-2)^2 = 13$ **19.** $(x-2)^2 + (y-3)^2 + (z+4)^2 = 25$

21.

$$x^2 + y^2 + z^2 + 4x - 8y - 2z + 5 = 0$$

$$x^2 + 4x + 4 + y^2 - 8y + 16 + z^2 - 2z + 1 = -5 + 4 + 16 + 1$$

$$(x+2)^2 + (y-4)^2 + (z-1)^2 = 16$$

center: $(-2, 4, 1)$, radius: 4

23.

$$2x^2+2y^2+2z^2+8x-4y=-1$$
$$2(x^2+4x+4)+2(y^2-2y+1)+2z^2=-1+8+2$$
$$(x+2)^2+(y-1)^2+z^2=\frac{9}{2}$$

center: $(-2,1,0)$, radius: $\frac{3}{2}\sqrt{2}$

25. $(2,3,-5)$ **27.** $(-2,3,5)$ **29.** $(-2,3,-5)$

31. $(-2,-3,-5)$ **33.** $(2,-5,5)$ **35.** $(-2,1,-3)$

37. Each such sphere has an equation of the form

$$(x-a)^2+(y-a)^2+(z-a)^2=a^2.$$

Substituting $x=5$, $y=1$, $z=4$ we get

$$(5-a)^2+(1-a)^2+(4-a)^2=a^2.$$

This reduces to $a^2-10a+21=0$ and gives $a=3$ or $a=7$. The equations are:

$$(x-3)^2+(y-3)^2+(z-3)^2=9;\quad (x-7)^2+(y-7)^2+(z-7)^2=49$$

39. Not a sphere; this equation is equivalent to:

$$(x-2)^2+(y+2)^2+(z+3)^2=-3$$

which has no (real) solutions.

41. $d(PR)=\sqrt{14}$, $d(QR)=\sqrt{45}$, $d(PQ)=\sqrt{59}$; $[d(PR)]^2+[d(QR)]^2=[d(PQ)]^2$

43. (a) Take R as (x,y,z). Since

$$d(P,R)=t\,d(P,Q)$$

we conclude by similar triangles that

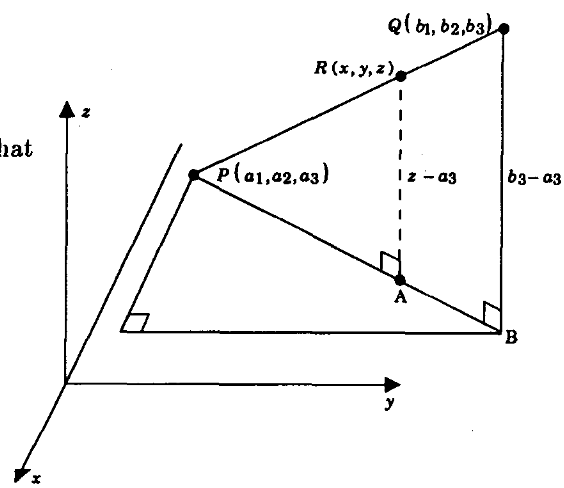
$$d(AR)=t\,d(B,Q)$$

and therefore

$$z-a_3=t(b_3-a_3).$$

Thus

$$z=a_3+t(b_3-a_3).$$



In similar fashion

$$x=a_1+t(b_1-a_1)\quad\text{and}\quad y=a_2+t(b_2-a_2).$$

(b) The midpoint of PQ, $\left(\frac{a_1+b_1}{2},\ \frac{a_2+b_2}{2},\ \frac{a_3+b_3}{2}\right)$, occurs at $t=\frac{1}{2}$.

SECTION 12.2

1. $\overrightarrow{PQ} = (3,4,-2)$; $\|\overrightarrow{PQ}\| = \sqrt{29}$

3. $\overrightarrow{PQ} = (-2,6)$; $\|\overrightarrow{PQ}\| = 2\sqrt{10}$

5. $\overrightarrow{PQ}(-4,2,2)$; $\|\overrightarrow{PQ}\| = 2\sqrt{6}$

7. $$2\mathbf{a} - \mathbf{b} = (2\cdot 1 - 3, 2\cdot[-2] - 0, 2\cdot 3 + 1) = (-1,-4,7)$$

9. $-2\mathbf{a} + \mathbf{b} - \mathbf{c} = [-2(\mathbf{a} - \mathbf{b})] - \mathbf{c} = (1+4, 4-2, -7-1) = (5,2,-8)$

11. $3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$

13. $-3\mathbf{i} - \mathbf{j} + 8\mathbf{k}$

15. 5

17. 3

19. $\sqrt{6}$

21. (a) $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{d}$ since $\mathbf{a} = \frac{1}{3}\mathbf{c} = -\frac{1}{2}\mathbf{d}$

(b) $\mathbf{a}$ and $\mathbf{c}$ since $\mathbf{a} = \frac{1}{3}\mathbf{c}$

(c) $\mathbf{a}$ and $\mathbf{c}$ both have direction opposite to $\mathbf{d}$

23. $\|\mathbf{a}\| = 5$; $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\dfrac{3}{5}, -\dfrac{4}{5}\right)$

25. $\|\mathbf{a}\| = 5$; $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \left(-\dfrac{4}{5}, 0, \dfrac{3}{5}\right)$

27. $\|\mathbf{a}\| = 3$; $\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{3}\mathbf{i} - \dfrac{2}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

29. $\|\mathbf{a}\| = \sqrt{14}$; $-\dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{1}{\sqrt{14}}\mathbf{i} - \dfrac{3}{\sqrt{14}}\mathbf{j} - \dfrac{2}{\sqrt{14}}\mathbf{k}$

31. (i) $\mathbf{a} + \mathbf{b}$ (ii) $-(\mathbf{a} + \mathbf{b})$ (iii) $\mathbf{a} - \mathbf{b}$ (iv) $\mathbf{b} - \mathbf{a}$

33. (a) $$\mathbf{a} - 3\mathbf{b} + 2\mathbf{c} + 4\mathbf{d} = (2\mathbf{i} - \mathbf{k}) - 3(\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + 2(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + 4(\mathbf{i} + \mathbf{j} + 6\mathbf{k})$$
$$= \mathbf{i} - 3\mathbf{j} + 10\mathbf{k}$$

(b) The vector equation

$$(1,1,6) = A(2,0,-1) + B(1,3,5) + C(-1,1,1)$$

implies

$$\begin{aligned} 1 &= 2A + B - C, \\ 1 &= 3B + C, \\ 6 &= -A + 5B + C. \end{aligned}$$

Simultaneous solution gives $A = -2$, $B = \frac{3}{2}$, $C = -\frac{7}{2}$.

35. $\|3\mathbf{i} + \mathbf{j}\| = \|\alpha\mathbf{j} - \mathbf{k}\| \implies 10 = \alpha^2 + 1$ so $\alpha = \pm 3$

37. $$\|\alpha\mathbf{i} + (\alpha - 1)\mathbf{j} + (\alpha + 1)\mathbf{k}\| = 2 \implies \alpha^2 + (\alpha-1)^2 + (\alpha+1)^2 = 4$$
$$\implies 3\alpha^2 = 2 \quad \text{so} \quad \alpha = \pm\tfrac{1}{3}\sqrt{6}$$

39. $\pm\frac{2}{13}\sqrt{13}(3\mathbf{j} + 2\mathbf{k})$ since $\|\alpha(3\mathbf{j} + 2\mathbf{k})\| = 2 \implies \alpha = \pm\frac{2}{13}\sqrt{13}$

41. $\mathbf{v} = (2\cos 30^\circ)\,\mathbf{i} + (2\sin 30^\circ)\,\mathbf{j} = \sqrt{3}\,\mathbf{i} + \mathbf{j}$

43. $\mathbf{v} = \cos(\pi/4)\,\mathbf{i} + \sin(\pi/4)\,\mathbf{j} = \dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}$

45. Since the $\mathbf{i}$ component is twice the $\mathbf{j}$ component, $\mathbf{v} = 2y\,\mathbf{i} + y\,\mathbf{j}$. Now, $\|\mathbf{v}\| = \sqrt{4y^2 + y^2} = 3$ which implies that $y = \dfrac{3}{\sqrt{5}}$. Thus, $\mathbf{v} = \dfrac{6}{\sqrt{5}}\mathbf{i} + \dfrac{3}{\sqrt{5}}\mathbf{j}$ or $\mathbf{v} = -\dfrac{6}{\sqrt{5}}\mathbf{i} - \dfrac{3}{\sqrt{5}}\mathbf{j}$.

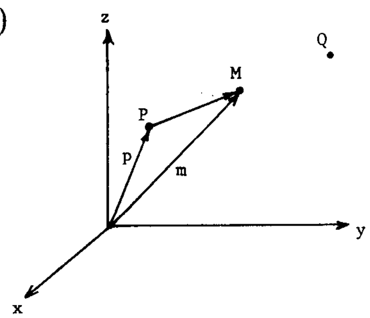
47. If $\mathbf{a}$ and $\mathbf{b}$ are the sides of a triangle, then $\mathbf{b} - \mathbf{a}$ is the third side. Now $\|\mathbf{a}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$, $\|\mathbf{b}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$, and $\|\mathbf{b} - \mathbf{a}\| = \sqrt{(1-2)^2 + (2+1)^2} = \sqrt{10}$. The triangle is a right triangle since $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{b} - \mathbf{a}\|^2$.

49. (a) Since $\|\mathbf{a} - \mathbf{b}\|$ and $\|\mathbf{a} + \mathbf{b}\|$ are the lengths of the diagonals of the parallelogram, the parallelogram must be a rectangle.

(b) Simplify

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2}.$$

51. (a)



(b) Let $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $M = (x_m, y_m, z_m)$. Then

$$\begin{aligned}(x_m, y_m, z_m) &= (x_1, y_1, z_1) + \tfrac{1}{2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)\\ &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)\end{aligned}$$

53. $\|\mathbf{F}_1\|\sin 40^\circ + \|\mathbf{F}_2\|\sin 25^\circ = 200$ and $\|\mathbf{F}_1\|\cos 40^\circ = \|\mathbf{F}_2\|\cos 25^\circ$

$\Longrightarrow \quad \|\mathbf{F}_1\| = 200.02$ and $\|\mathbf{F}_2\| = 169.05$

$\mathbf{F}_1 = -\|\mathbf{F}_1\|\cos 40^\circ\,\mathbf{i} + \|\mathbf{F}_1\|\sin 40^\circ\,\mathbf{j} = -153.21\,\mathbf{i} + 128.56\,\mathbf{j}$

$\mathbf{F}_2 = \|\mathbf{F}_2\|\cos 25^\circ\,\mathbf{i} + \|\mathbf{F}_2\|\sin 25^\circ\,\mathbf{j} = 153.21\,\mathbf{i} + 71.44\,\mathbf{j}$

55. $\mathbf{V}_1 = 600\sin 30^\circ\,\mathbf{i} + 600\cos 30^\circ\,\mathbf{j} = 300\,\mathbf{i} + 300\sqrt{3}\,\mathbf{j}$ and

$\mathbf{V}_2 = 50\sin 45^\circ\,\mathbf{i} - 50\cos 45^\circ\,\mathbf{j} = 25\sqrt{2}\,\mathbf{i} - 25\sqrt{2}\,\mathbf{j}$

$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = (300 + 25\sqrt{2})\,\mathbf{i} + (300\sqrt{3} - 25\sqrt{2})\,\mathbf{j} \cong 335.36\,\mathbf{i} + 484.26\,\mathbf{j}$

true course: $\theta = \tan^{-1}\dfrac{335.36}{484.26} = 34.70^\circ$; or $N\,34.70^\circ\,E$.

ground speed: $\|\mathbf{V}\| = \sqrt{(335.56)^2 + (484.26)^2} \cong 589.05$ mi/hr

57. (a) $\|\mathbf{r} - \mathbf{a}\| = 3$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(b) $\|\mathbf{r}\| \le 2$ (c) $\|\mathbf{r} - \mathbf{a}\| \le 1$ where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

(d) $\|\mathbf{r} - \mathbf{a}\| = \|\mathbf{r} - \mathbf{b}\|$ (e) $\|\mathbf{r} - \mathbf{a}\| + \|\mathbf{r} - \mathbf{b}\| = k$

SECTION 12.3

1. $\mathbf{a}\cdot\mathbf{b}=(2)(-2)+(-3)(0)+(1)(3)=-1$

3. $\mathbf{a}\cdot\mathbf{b}=(2)(1)+(-4)(1/2)=0$

5. $\mathbf{a}\cdot\mathbf{b}=(2)(1)+(1)(1)-(2)(2)=-1$

7. $\mathbf{a}\cdot\mathbf{b}$

9. $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c}+\mathbf{b}\cdot(\mathbf{c}+\mathbf{a})=\mathbf{a}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$

11. (a) $\mathbf{a}\cdot\mathbf{b}=(2)(3)+(1)(-1)+(0)(2)=5$

$\mathbf{a}\cdot\mathbf{c}=(2)(4)+(1)(0)+(0)(3)=8$

$\mathbf{b}\cdot\mathbf{c}=(3)(4)+(-1)(0)+(2)(3)=18$

(b) $\|\mathbf{a}\|=\sqrt{5},\quad \|\mathbf{b}\|=\sqrt{14},\quad \|\mathbf{c}\|=5.$ Then,

$$\cos \measuredangle(\mathbf{a},\mathbf{b})=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}=\frac{5}{(\sqrt{5})\,(\sqrt{14})}=\frac{1}{14}\sqrt{70},$$

$$\cos \measuredangle(\mathbf{a},\mathbf{c})=\frac{8}{(\sqrt{5})\,(5)}=\frac{8}{25}\sqrt{5},$$

$$\cos \measuredangle(\mathbf{b},\mathbf{c})=\frac{18}{(\sqrt{14})\,(5)}=\frac{9}{35}\sqrt{14}.$$

(c) $\mathbf{u_b}=\dfrac{1}{\sqrt{14}}(3\mathbf{i}-\mathbf{j}+2\mathbf{k}),\quad \text{comp}_\mathbf{b}\,\mathbf{a}=\mathbf{a}\cdot\mathbf{u_b}=\dfrac{1}{\sqrt{14}}(6-1)=\dfrac{5}{14}\sqrt{14},$

$\mathbf{u_c}=\frac{1}{5}(4\mathbf{i}+3\mathbf{k}),\quad \text{comp}_\mathbf{c}\,\mathbf{a}=\mathbf{a}\cdot\mathbf{u_c}=\frac{8}{5}$

(d) $\mathbf{proj_b\,a}=(\text{comp}_\mathbf{b}\,\mathbf{a})\,\mathbf{u_b}=\frac{5}{14}(3\mathbf{i}-\mathbf{j}+2\mathbf{k})$

$\mathbf{proj_c\,a}=(\text{comp}_\mathbf{c}\,\mathbf{a})\,\mathbf{u_c}=\frac{8}{25}(4\mathbf{i}+3\mathbf{k})$

13. $\mathbf{u}=\cos\dfrac{\pi}{3}\mathbf{i}+\cos\dfrac{\pi}{4}\mathbf{j}+\cos\dfrac{2\pi}{3}\mathbf{k}=\dfrac{1}{2}\mathbf{i}+\dfrac{1}{2}\sqrt{2}\mathbf{j}-\dfrac{1}{2}\mathbf{k}$

15. $\cos\theta=\dfrac{(3\mathbf{i}-\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-3\mathbf{k})}{\|3\mathbf{i}-\mathbf{j}-2\mathbf{k}\|\,\|\mathbf{i}+2\mathbf{j}-3\mathbf{k}\|}=\dfrac{7}{\sqrt{14}\sqrt{14}}=\dfrac{1}{2},\quad \theta=\dfrac{\pi}{3}$

17. Since $\|\mathbf{i}-\mathbf{j}+\sqrt{2}\,\mathbf{k}\|=2$, we have $\cos\alpha=\frac{1}{2},\ \cos\beta=-\frac{1}{2},\ \cos\gamma=\frac{1}{2}\sqrt{2}.$
The direction angles are $\frac{1}{3}\pi,\ \frac{2}{3}\pi,\ \frac{1}{4}\pi.$

19. $\theta=\cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\left(\dfrac{-1}{\sqrt{231}}\right)\cong 2.2$ radians or 126.3°

21. $\theta=\cos^{-1}\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}=\cos^{-1}\left(\dfrac{-13}{5\sqrt{10}}\right)\cong 2.5$ radians or 145.3°

23. $\|\mathbf{a}\|=\sqrt{1^2+2^2+2^2}=3;$ $\quad\cos\alpha=\dfrac{1}{3},\ \cos\beta=\dfrac{2}{3},\ \cos\gamma=\dfrac{2}{3}$

$\alpha\cong 70.5^\circ\quad \beta\cong 48.2^\circ,\quad \gamma\cong 48.2^\circ$

25. $\|\mathbf{a}\|=\sqrt{3^2+(12)^2+4^2}=13;$ $\quad\cos\alpha=\dfrac{3}{13},\ \cos\beta=\dfrac{12}{13}\ \ \cos\gamma=\dfrac{4}{13}$

$\alpha\cong 76.7^\circ\quad \beta\cong 22.6^\circ,\quad \gamma\cong 72.1^\circ$

27. (a) $\mathbf{proj_b}\,\alpha\mathbf{a} = (\alpha\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b} = \alpha(\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b} = \alpha\,\mathbf{proj_b}\,\mathbf{a}$

(b)
$$\begin{aligned}\mathbf{proj_b}\,(\mathbf{a}+\mathbf{c}) &= [(\mathbf{a}+\mathbf{c})\cdot\mathbf{u_b}]\,\mathbf{u_b}\\ &= (\mathbf{a}\cdot\mathbf{u_b}+\mathbf{c}\cdot\mathbf{u_b})\mathbf{u_b}\\ &= (\mathbf{a}\cdot\mathbf{u_b})\mathbf{u_b}+(\mathbf{c}\cdot\mathbf{u_b})\mathbf{u_b} = \mathbf{proj_b}\,\mathbf{a}+\mathbf{proj_b}\,\mathbf{c}\end{aligned}$$

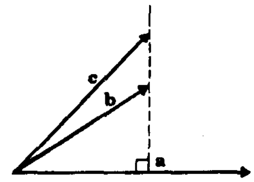
29. (a) For $\mathbf{a}\neq\mathbf{0}$ the following statements are equivalent:

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c},\quad \mathbf{b}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{a},$$

$$\mathbf{b}\cdot\frac{\mathbf{a}}{\|\mathbf{a}\|}=\mathbf{c}\cdot\frac{\mathbf{a}}{\|\mathbf{a}\|},\quad \mathbf{b}\cdot\mathbf{u_a}=\mathbf{c}\cdot\mathbf{u_a}$$

$$(\mathbf{b}\cdot\mathbf{u_a})\mathbf{u_a}=(\mathbf{c}\cdot\mathbf{u_a})\mathbf{u_a},$$

$$\mathbf{proj_a}\,\mathbf{b}=\mathbf{proj_a}\,\mathbf{c}.$$



$$\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}\quad\text{but}\quad\mathbf{b}\neq\mathbf{c}$$

(b) $\mathbf{b} \underset{(12.3.13)}{=} (\mathbf{b}\cdot\mathbf{i})\mathbf{i}+(\mathbf{b}\cdot\mathbf{j})\mathbf{j}+(\mathbf{b}\cdot\mathbf{k})\mathbf{k} = (\mathbf{c}\cdot\mathbf{i})\mathbf{i}+(\mathbf{c}\cdot\mathbf{j})\mathbf{j}+(\mathbf{c}\cdot\mathbf{k})\mathbf{k} \underset{(12.3.13)}{=} \mathbf{c}$

31. (a)
$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2-\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})-(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &= [(\mathbf{a}\cdot\mathbf{a})+2(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b})]-[(\mathbf{a}\cdot\mathbf{a})-2(\mathbf{a}\cdot\mathbf{b})+(\mathbf{b}\cdot\mathbf{b})]=4(\mathbf{a}\cdot\mathbf{b})\end{aligned}$$

(b) The following statements are equivalent:

$$\mathbf{a}\perp\mathbf{b},\quad \mathbf{a}\cdot\mathbf{b}=0,\quad \|\mathbf{a}+\mathbf{b}\|^2-\|\mathbf{a}-\mathbf{b}\|^2=0,\quad \|\mathbf{a}+\mathbf{b}\|=\|\mathbf{a}-\mathbf{b}\|.$$

(c) By (b), the relation $\|\mathbf{a}+\mathbf{b}\|=\|\mathbf{a}-\mathbf{b}\|$ gives $\mathbf{a}\perp\mathbf{b}$. The relation $\mathbf{a}+\mathbf{b}\perp\mathbf{a}-\mathbf{b}$ gives

$$0=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\|\mathbf{a}\|^2-\|\mathbf{b}\|^2\quad\text{and thus}\quad\|\mathbf{a}\|=\|\mathbf{b}\|.$$

The parallelogram is a square since it has two adjacent sides of equal length and these meet at right angles.

33. $\|\mathbf{a}+\mathbf{b}\|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=\mathbf{a}\cdot\mathbf{a}+2\,\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}=\|\mathbf{a}\|^2+2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

$\|\mathbf{a}-\mathbf{b}\|^2=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\mathbf{a}\cdot\mathbf{a}-2\,\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}=\|\mathbf{a}\|^2-2\mathbf{a}\cdot\mathbf{b}+\|\mathbf{b}\|^2$

Add the two equations and the result follows.

35. Let $\theta_1,\theta_2,\theta_3$ be the direction angles of $-\mathbf{a}$. Then

$$\theta_1=\cos^{-1}\left[\frac{(-\mathbf{a}\cdot\mathbf{i})}{\|-\mathbf{a}\|}\right]=\cos^{-1}\left[-\frac{(\mathbf{a}\cdot\mathbf{i})}{\|\mathbf{a}\|}\right]=\cos^{-1}(-\cos\alpha)=\cos^{-1}(\pi-\alpha)=\pi-\alpha.$$

Similarly $\theta_2=\pi-\beta$ and $\theta_3=\pi-\gamma$.

37. If $\mathbf{a}\perp\mathbf{b}$ and $\mathbf{a}\perp\mathbf{c}$, then

$$\mathbf{a}\cdot\mathbf{b}=0,\quad\mathbf{a}\cdot\mathbf{c}=0$$

$$\mathbf{a}\cdot(\alpha\mathbf{b}+\beta\mathbf{c})=\alpha(\mathbf{a}\cdot\mathbf{b})+\beta(\mathbf{a}\cdot\mathbf{c})=0$$

$$\mathbf{a}\perp(\alpha\mathbf{b}+\beta\mathbf{c}).$$

39. Existence of decomposition:

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{u_b})\mathbf{u_b} + [\mathbf{a} - (\mathbf{a} \cdot \mathbf{u_b})\mathbf{u_b}].$$

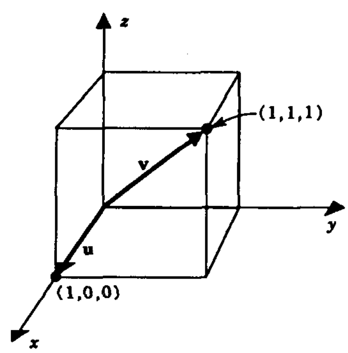
Uniqueness of decomposition: suppose that

$$\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}.$$

Then the vector $\mathbf{a}_{\parallel} - \mathbf{A}_{\parallel} = \mathbf{A}_{\perp} - \mathbf{a}_{\perp}$ is both parallel to $\mathbf{b}$ and perpendicular to $\mathbf{b}$. (Exercises 37 and 38.) Therefore it is zero. Consequently $\mathbf{A}_{\parallel} = \mathbf{a}_{\parallel}$ and $\mathbf{A}_{\perp} = \mathbf{a}_{\perp}$.

41. $\cos\dfrac{\pi}{3} = \dfrac{\mathbf{c}\cdot\mathbf{d}}{\|\mathbf{c}\|\,\|\mathbf{d}\|},\quad \dfrac{1}{2} = \dfrac{2x+1}{x^2+2},\quad x^2 = 4x;\qquad x = 0, 4$

43.



We take $\mathbf{u} = \mathbf{i}$ as an edge and $\mathbf{v} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ as a diagonal of a cube. Then,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{1}{3}\sqrt{3},$$

$$\theta = \cos^{-1}\left(\tfrac{1}{3}\sqrt{3}\right) \cong 0.96 \text{ radians.}$$

45. (a) The direction angles of a vector always satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

and, as you can check,

$$\cos^2 \tfrac{1}{4}\pi + \cos^2 \tfrac{1}{6}\pi + \cos^2 \tfrac{2}{3}\pi \neq 1.$$

(b) The relation

$$\cos^2\alpha + \cos^2 \tfrac{1}{4}\pi + \cos^2 \tfrac{1}{4}\pi = 1$$

gives

$$\cos^2\alpha + \tfrac{1}{2} + \tfrac{1}{2} = 1, \quad \cos\alpha = 0, \quad a_1 = \|\mathbf{a}\|\cos\alpha = 0.$$

47. Set $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. The relations

$$(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}+\mathbf{k}) = 0 \quad\text{and}\quad (a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\cdot(3\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = 0$$

give

$$a + 2b + c = 0 \qquad 3a - 4b + 2c = 0$$

so that $b = \frac{1}{8}a$ and $c = -\frac{5}{4}a$.

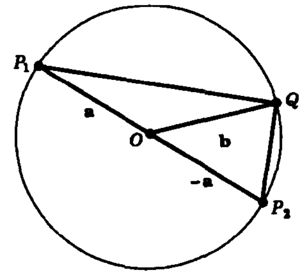
Then, since $\mathbf{u}$ is a unit vector,

$$a^2 + b^2 + c^2 = 1, \quad a^2 + \left(\frac{a}{8}\right)^2 + \left(\frac{-5a}{4}\right)^2 = 1, \quad \frac{165}{64}a^2 = 1.$$

Thus, $a = \pm\dfrac{8}{165}\sqrt{165}$ and $\mathbf{u} = \pm\dfrac{\sqrt{165}}{165}(8\mathbf{i}+\mathbf{j}-10\mathbf{k})$.

49. Place center of sphere at the origin.

$$\overrightarrow{P_1Q} \cdot \overrightarrow{P_2Q} = (-\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b})$$
$$= -\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$$
$$= 0.$$



51. (a) $W = \mathbf{F} \cdot \mathbf{r}$ (b) 0 (c) $\|\mathbf{F}\|\mathbf{i} \cdot (b-a)\mathbf{i} = \|\mathbf{F}\|(b-a)$

53. (a) $W = (15\cos 35^\circ\,\mathbf{i} + 15\sin 35^\circ\,\mathbf{j}) \cdot (50\,\mathbf{i}) = 15 \cdot \cos 35^\circ \cdot 50 = 614.36$ joules

(b) $W = (15\cos 50^\circ\,\mathbf{i} + 15\sin 50^\circ\,\mathbf{j}) \cdot (50\cos 15^\circ\,\mathbf{i} + 50\sin 15^\circ\,\mathbf{j})$

$= 15 \cdot 50(\cos 50^\circ \cos 15^\circ + \sin 50^\circ \sin 15^\circ = 15 \cdot 50\cos 35^\circ = 614.36$ joules

55. Let $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = C$.

(a) $W_1 = C\cos\theta_1$; $W_2 = C\cos\theta_2 = C\cos(-\theta_1) = C\cos\theta_1 = W_1$ Thus, $W_1 = W_2$

(b) $W_1 = C \cdot \cos(\pi/3) \cdot \|\mathbf{r}\| = \frac{1}{2}C\|\mathbf{r}\|$ and $W_2 = C \cdot \cos(\pi/6) \cdot \|\mathbf{r}\| = \frac{\sqrt{3}}{2}C\|\mathbf{r}\|$

Thus, $W_2 = \sqrt{3}\,W_1$

SECTION 12.4

1. $(\mathbf{i}+\mathbf{j}) \times (\mathbf{i}-\mathbf{j}) = [\mathbf{i} \times (\mathbf{i}-\mathbf{j})] + [\mathbf{j} \times (\mathbf{i}-\mathbf{j})] = (0-\mathbf{k}) + (-\mathbf{k}-0) = -2\mathbf{k}$

3. $(\mathbf{i}-\mathbf{j}) \times (\mathbf{j}-\mathbf{k}) = [\mathbf{i} \times (\mathbf{j}-\mathbf{k})] - [\mathbf{j} \times (\mathbf{j}-\mathbf{k})] = (\mathbf{j}+\mathbf{k}) - (0-\mathbf{i}) = \mathbf{i}+\mathbf{j}+\mathbf{k}$

5. $(2\mathbf{j}-\mathbf{k}) \times (\mathbf{i}-3\mathbf{j}) = [2\mathbf{j} \times (\mathbf{i}-3\mathbf{j})] - [\mathbf{k} \times (\mathbf{i}-3\mathbf{j})] = (-2\mathbf{k}) - (\mathbf{j}+3\mathbf{i}) = -3\mathbf{i}-\mathbf{j}-2\mathbf{k}$

or

$$(2\mathbf{j}-\mathbf{k}) \times (\mathbf{i}-3\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ 1 & -3 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & -1 \\ 1 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} = -3\mathbf{i}-\mathbf{j}-2\mathbf{k}$$

7. $\mathbf{j} \cdot (\mathbf{i} \times \mathbf{k}) = \mathbf{j} \cdot (-\mathbf{j}) = -1$ **9.** $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$ **11.** $\mathbf{j} \cdot (\mathbf{k} \times \mathbf{i}) = \mathbf{j} \cdot (\mathbf{j}) = 1$

13. $$(\mathbf{i}+3\mathbf{j}-\mathbf{k}) \times (\mathbf{i}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 0 & 1 \end{vmatrix} = [(3)(1)-(-1)(0)]\mathbf{i} - [(1)(1)-(-1)(1)]\mathbf{j} + [(1)0-(3)(1)]\mathbf{k}$$
$$= 3\mathbf{i}-2\mathbf{j}-3\mathbf{k}$$

15. $$(\mathbf{i}+\mathbf{j}+\mathbf{k}) \times (2\mathbf{i}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = [(1)(1)-(1)(0)]\mathbf{i} - [(1)(1)-(1)(2)]\mathbf{j} + [(1)(0)-(1)(2)]\mathbf{k}$$
$$= \mathbf{i}+\mathbf{j}-2\mathbf{k}$$

17. $[2\mathbf{i}+\mathbf{j}]\cdot[(\mathbf{i}-3\mathbf{j}+\mathbf{k})\times(4\mathbf{i}+\mathbf{k})]=\begin{vmatrix}1 & -3 & 1\\ 4 & 0 & 1\\ 2 & 1 & 0\end{vmatrix}=$

$$[(0)(0)-(1)(1)]-(-3)[(4)(0)-(1)(2)]+[(4)(1)-(0)(2)]=-3$$

19.

$$\begin{aligned}[(\mathbf{i}-\mathbf{j})\times(\mathbf{j}-\mathbf{k})]\times[\mathbf{i}+5\mathbf{k}] &= \{[\mathbf{i}\times(\mathbf{j}-\mathbf{k})]-[\mathbf{j}\times(\mathbf{j}-\mathbf{k})]\}\times[\mathbf{i}+5\mathbf{k}]\\ &= [(\mathbf{k}+\mathbf{j})-(-\mathbf{i})]\times[\mathbf{i}+5\mathbf{k}]\\ &= (\mathbf{i}+\mathbf{j}+\mathbf{k})\times(\mathbf{i}+5\mathbf{k})\\ &= [(\mathbf{i}+\mathbf{j}+\mathbf{k})\times\mathbf{i}]\ +\ [(\mathbf{i}+\mathbf{j}+\mathbf{k})\times 5\mathbf{k}]\\ &= (-\mathbf{k}+\mathbf{j})+(-5\mathbf{j}+5\mathbf{i})\\ &= 5\mathbf{i}-4\mathbf{j}-\mathbf{k}\end{aligned}$$

21. $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}1 & -3 & 1\\ 4 & 0 & 1\\ 2 & 1 & 0\end{vmatrix}=3\mathbf{i}-3\mathbf{j}-6\mathbf{k}$

$\dfrac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}=\dfrac{1}{\sqrt{6}}\mathbf{i}-\dfrac{1}{\sqrt{6}}\mathbf{j}-\dfrac{2}{\sqrt{6}}\mathbf{k};\quad \dfrac{\mathbf{b}\times\mathbf{a}}{\|\mathbf{b}\times\mathbf{a}\|}=-\dfrac{1}{\sqrt{6}}\mathbf{i}+\dfrac{1}{\sqrt{6}}\mathbf{j}+\dfrac{2}{\sqrt{6}}\mathbf{k}$

23. Set $\mathbf{a}=\overrightarrow{PQ}=-\mathbf{i}+2\mathbf{k}$ and $\mathbf{b}=\overrightarrow{PR}=2\mathbf{i}-\mathbf{k}$. Then

$$\mathbf{N}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 0 & 2\\ 2 & 0 & -1\end{vmatrix}=3\mathbf{j}$$

and $A=\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|=\frac{1}{2}\|3\mathbf{j}\|=\frac{3}{2}$.

25. Set $\mathbf{a}=\overrightarrow{PQ}=\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{b}=\overrightarrow{PR}=-\mathbf{i}+3\mathbf{j}-\mathbf{k}$. Then

$$\mathbf{N}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}=8\mathbf{j}+4\mathbf{j}+4\mathbf{k}$$

and $A=\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|=\frac{1}{2}\|8\mathbf{i}+4\mathbf{j}+4\mathbf{k}\|=\frac{1}{2}\sqrt{8^2+4^2+4^2}=2\sqrt{6}$.

27. $V=\left|[(\mathbf{i}+\mathbf{j})\times(2\mathbf{i}-\mathbf{k})]\cdot(3\mathbf{j}+\mathbf{k})\right|=\left|(-\mathbf{i}+\mathbf{j}-2\mathbf{k})\cdot(3\mathbf{j}+\mathbf{k})\right|=1$

29. $V=\overrightarrow{OP}\cdot\left(\overrightarrow{OQ}\times\overrightarrow{OR}\right)=\begin{vmatrix}1 & 2 & 3\\ 1 & 1 & 2\\ 2 & 1 & 1\end{vmatrix}=2$

31.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= [\mathbf{a}\times(\mathbf{a}-\mathbf{b})]+[\mathbf{b}\times(\mathbf{a}-\mathbf{b})]\\ &= [\mathbf{a}\times(-\mathbf{b})]+[\mathbf{b}\times\mathbf{a}]\\ &= -(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times\mathbf{b})=-2(\mathbf{a}\times\mathbf{b})\end{aligned}$$

33. $\mathbf{a} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{a} \times \mathbf{j} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{a} \| \mathbf{i}$ and $\mathbf{a} \| \mathbf{j} \quad \Longrightarrow \quad \mathbf{a} = \mathbf{0}$

35.
$$\begin{aligned}(\alpha\mathbf{a} + \beta\mathbf{b}) \times (\gamma\mathbf{a} + \delta\mathbf{b}) &= (\alpha\mathbf{a} \times \delta\mathbf{b}) + (\beta\mathbf{b} \times \gamma\mathbf{a}) \\ &= \alpha\delta(\mathbf{a} \times \mathbf{b}) - \beta\gamma(\mathbf{a} \times \mathbf{b}) \\ &= (\alpha\delta - \beta\gamma)(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} (\mathbf{a} \times \mathbf{b})\end{aligned}$$

37. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{a} \times -\mathbf{c}) \cdot \mathbf{b}$

$\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = (-\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

39. $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane determined by $\mathbf{a}$ and $\mathbf{b}$;

$\mathbf{c}$ is in this plane iff $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = 0$.

41. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \quad \Longrightarrow \quad \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$; $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$.

$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \quad \Longrightarrow \quad \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$; $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$.

Since $\mathbf{a} \neq \mathbf{0}$ it follows that $\mathbf{b} - \mathbf{c} = \mathbf{0}$ or $\mathbf{b} = \mathbf{c}$.

43. $\mathbf{c} \times \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{a} \underset{\text{Exercise 42(a)}}{=} (\mathbf{a} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a} \underset{\mathbf{a} \cdot \mathbf{b} = 0}{=} (\mathbf{a} \cdot \mathbf{a})\mathbf{b} = \|\mathbf{a}\|^2\mathbf{b}$

45. Expanding the determinant by the bottom row gives
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

47. $\|\boldsymbol{\tau}\| = \|\mathbf{r}\| \cdot \|\mathbf{F}\| \sin\theta = (10)(20)\sin 50^\circ = 153.21$ inch-lb $= 12.77$ ft-lb;

the bolt moves into the plane of the paper.

SECTION 12.5

1. P (when $t = 0$) and Q (when $t = -1$)

3. Take $\mathbf{r}_0 = \overrightarrow{OP} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{d} = \mathbf{k}$. Then, $\mathbf{r}(t) = (3\mathbf{i} + \mathbf{j}) + t\mathbf{k}$.

5. Take $\mathbf{r}_0 = \mathbf{0}$ and $\mathbf{d} = \overrightarrow{OQ}$. Then, $\mathbf{r}(t) = t(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$.

7. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ so direction numbers are $1, -1, 1$. Using P as a point on the line, we have

$$x(t) = 1 + t, \quad y(t) = -t, \quad z(t) = 3 + t.$$

9. The line is parallel to the y-axis so we can take $0, 1, 0$ as direction numbers. Therefore

$$x(t) = 2, \quad y(t) = -2 + t, \quad z(t) = 3.$$

11. Since the line $2(x+1) = 4(y-3) = z$ can be written

$$\frac{x+1}{2} = \frac{y-3}{1} = \frac{z}{4},$$

it has direction numbers $2, 1, 4$. The line through $P(-1, 2, -3)$ with direction vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ can be parametrized

$$\mathbf{r}(t) = (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}).$$

13. We set $\mathbf{r}_1(t) = \mathbf{r}_2(u)$ and solve for t and u:

$$\mathbf{i} + t\mathbf{j} = \mathbf{j} + u(\mathbf{i} + \mathbf{j}),$$

$$(1-u)\,\mathbf{i} + (-1 - u + t)\,\mathbf{j} = \mathbf{0}.$$

Thus,

$$1 - u = 0 \quad \text{and} \quad -1 - u + t = 0.$$

The equation gives $u = 1$, $t = 2$. The point of intersection is $P(1, 2, 0)$.

As direction vectors for the lines we can take $\mathbf{u} = \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j}$. Thus

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{1}{(1)(\sqrt{2})} = \frac{1}{2}\sqrt{2}.$$

The angle of intersection is $\frac{1}{4}\pi$ radians.

15. We solve the system

$$3 + t = 1, \quad 1 - t = 4 + u, \quad 5 + 2t = 2 + u$$

for t and u to find that $t = -2$, $u = -1$. The point of intersection is $(1, 3, 1)$.

Since $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is a direction vector for l_1 and $\mathbf{j} + \mathbf{k}$ is a direction vector for l_2,

$$\cos\theta = \frac{(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{j} + \mathbf{k})}{\sqrt{6}\sqrt{2}} = \frac{1}{2\sqrt{3}} = \frac{1}{6}\sqrt{3} \quad \text{and} \quad \theta \cong 1.28 \text{ radians.}$$

17. $\left(x_0 - \dfrac{d_1}{d_3}z_0,\ \ y_0 - \dfrac{d_2}{d_3}z_0,\ \ 0\right)$

19. The lines are parallel.

21. $\mathbf{r}(t) = (2\mathbf{i} + 7\mathbf{j} - \mathbf{k}) + t(2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}), \quad 0 \le t \le 1$

23. Set

$$\mathbf{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}}{\|-4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}\|} = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

Then $\mathbf{r}(t) = (6\mathbf{i} - 5\mathbf{j} + \mathbf{k}) + t\mathbf{u}$ is $\overrightarrow{OP}$ at $t = 9$ and it is $\overrightarrow{OQ}$ at $t = 15$. (Check this.)

Answer: $\mathbf{u} = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}, \quad 9 \leq t \leq 15.$

25. The given line, call it l, has direction vector $2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$.

If $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a direction vector for a line perpendicular to l, then

$$(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2a - 4b + 6c = 0.$$

The lines through $P(3, -1, 8)$ perpendicular to l can be parametrized

$$X(u) = 3 + au, \quad Y(u) = -1 + bu, \quad Z(u) = 8 + cu$$

with $2a - 4b + 6c = 0$.

27. $d(P, l) = \dfrac{\|(\mathbf{i} + 2\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\|}{\|2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\|} = 1$

29. The line contains the point $P_0(1, 0, 2)$. Therefore

$$d(P, l) = \frac{\|(2\mathbf{j} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\|}{\|\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\|} = \sqrt{\frac{69}{14}} \cong 2.22$$

31. The line contains the point $P_0(2, -1, 0)$. Therefore

$$d(P, l) = \frac{\|(\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (\mathbf{i} + \mathbf{j})\|}{\|\mathbf{i} + \mathbf{j}\|} = \sqrt{3} \cong 1.73.$$

33. (a) The line passes through $P(1, 1, 1)$ with direction vector $\mathbf{i} + \mathbf{j}$. Therefore

$$d(0, l) = \frac{\|(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + \mathbf{j})\|}{\|\mathbf{i} + \mathbf{j}\|} = 1.$$

(b) The distance from the origin to the line segment is $\sqrt{3}$.

Solution. The line segment can be parametrized

$$\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j}), \quad t \in [0, 1].$$

This is the set of all points $P(1 + t, 1 + t, 1)$ with $t \in [0, 1]$.

The distance from the origin to such a point is

$$f(t) = \sqrt{2\,(1 + t^2) + 1}.$$

The minimum value of this function is $f(0) = \sqrt{3}$.

Explanation. The point on the line through P and Q closest to the origin is not on the line segment $\overline{PQ}$.

35. We begin with $\mathbf{r}(t) = \mathbf{j} - 2\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. The scalar t_0 for which $\mathbf{r}(t_0) \perp l$ can be found by solving the equation

$$[\mathbf{j} - 2\mathbf{k} + t_0(\mathbf{i} - \mathbf{j} + 3\mathbf{k})] \cdot [\mathbf{i} - \mathbf{j} + 3\mathbf{k}] = 0.$$

This equation gives $-7 + 11t_0 = 0$ and thus $t_0 = 7/11$. Therefore

$$\mathbf{r}(t_0) = \mathbf{j} - 2\mathbf{k} + \tfrac{7}{11}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \tfrac{7}{11}\mathbf{i} + \tfrac{4}{11}\mathbf{j} - \tfrac{1}{11}\mathbf{k}.$$

The vectors of norm 1 parallel to $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ are

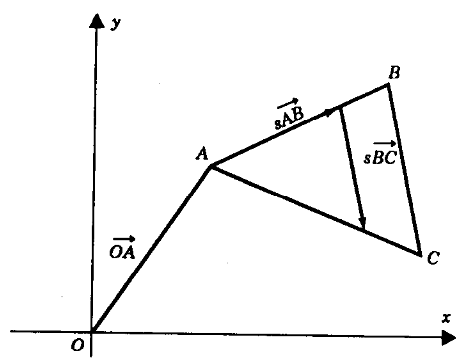
$$\pm\frac{1}{\sqrt{11}}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}).$$

The standard parametrizations are

$$\begin{aligned}\mathbf{R}(t) &= \frac{7}{11}\mathbf{i} + \frac{4}{11}\mathbf{j} - \frac{1}{11}\mathbf{k} \pm \frac{t}{\sqrt{11}}(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \\ &= \frac{1}{11}(7\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \pm t\left[\frac{\sqrt{11}}{11}(\mathbf{i} - \mathbf{j} + 3\mathbf{k})\right].\end{aligned}$$

37. $0 < t < s$

By similar triangles, if $0 < s < 1$, the tip of $\overrightarrow{OA} + s\overrightarrow{AB} + s\overrightarrow{BC}$ falls on $\overline{AC}$. If $0 < t < s$, then the tip of $\overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{BC}$ falls short of $\overline{AC}$ and stays within the triangle. Clearly all points in the interior of the triangle can be reached in this manner.



SECTION 12.6

1. Q

3. Since $\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ is normal to the plane, we have

$$(x - 2) - 4(y - 3) + 3(z - 4) = 0 \quad \text{and thus} \quad x - 4y + 3z - 2 = 0.$$

5. The vector $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ is normal to the given plane and thus to every parallel plane: the equation we want can be written

$$3(x - 2) - 2(y - 1) + 5(z - 1) = 0, \quad 3x - 2y + 5z - 9 = 0.$$